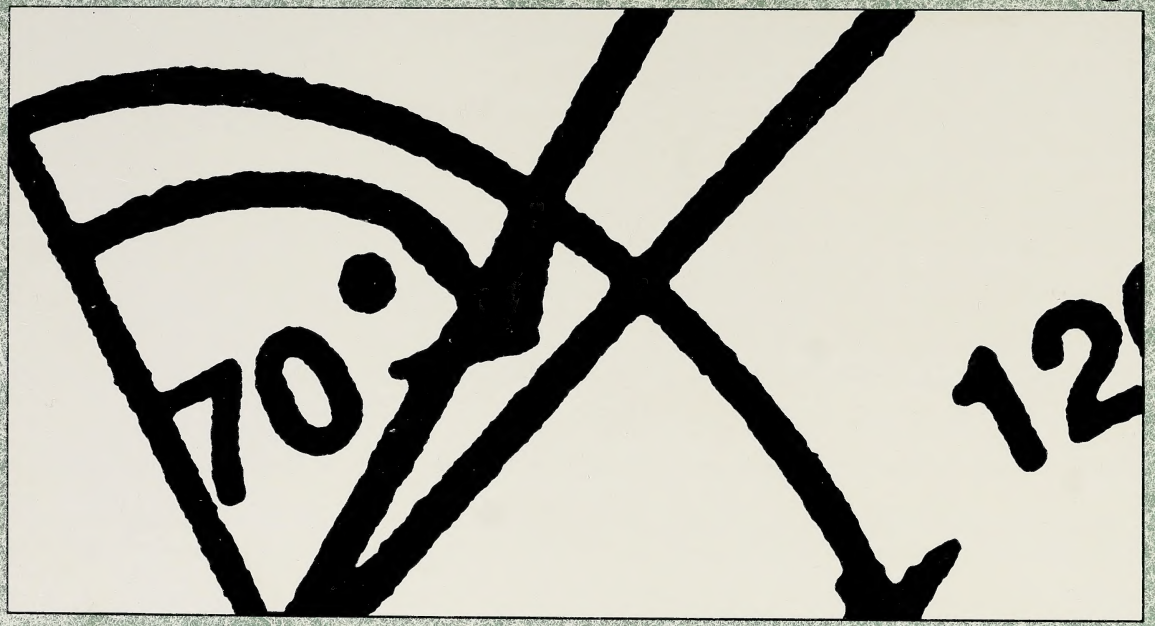
MATHEMATICS 31



Distance Learning

UNIT 5: GEOMETRIC VECTORS AND THEIR APPLICATION

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Welcome

Distance Learning

You have chosen an alternate form of learning that allows you to work at your own pace. You will be responsible for your own schedule, for disciplining yourself to study the units thoroughly, and for completing your units regularly. We wish you much success and enjoyment in your studies.

Mathematics 31 Student Module Unit 5 Geometric Vectors and Their Application Alberta Distance Learning Centre ISBN No. 0-7741-0335-3

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General Information

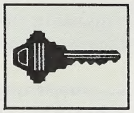
This information explains the basic layout of each booklet.

- **What You Already Know** and **Review** are to help you look back at what you have previously studied. The questions are to jog your memory and to prepare you for the learning that is going to happen in this unit.
- As you begin each **Topic**, spend a little time looking over the components. Doing this will give you a preview of what will be covered in the topic and will set your mind in the direction of learning.
- **Exploring the Topic** includes the objectives, concept development, and activities for each objective. Use your own papers to arrive at the answers in the activities.
- **Extra Help** reviews the topic. If you had any difficulty with **Exploring the Topic**, you may find this part helpful.
- **Extensions** gives you the opportunity to take the topic one step further.
- To summarize what you have learned, and to find instructions on doing the unit assignment, turn to the **Unit Summary** at the end of the unit.
- The **Appendices** include the solutions to **Activities (Appendix A)** and any other charts, tables, etc. which may be referred to in the topics **(Appendix B, etc.)**.

Visual Cues

Visual cues are pictures that are used to identify important areas of the material. They are found throughout the booklet.

An explanation of what they mean is written beside each visual cue.

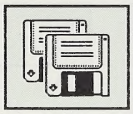


Key Idea
- flagging important ideas

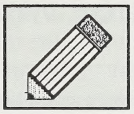
Audiotape
- learning by listening to an audiotape

What You Already Know
- reviewing what you already know

Another View
- exploring different perspectives



Computer Software
- learning by using computer software



Review
- studying previous concepts

Solutions
- correcting the activities

Videotape
- learning by viewing a videotape

Introduction
- introducing the unit

Extra Help
- providing additional study

Print Pathway
- choosing a print alternative

What Lies Ahead
- previewing the unit

Extensions
- going on with the topic

Calculator
- using your calculator

Exploring the Topic
- actively learning new concepts

What You Have Learned
- summarizing what you have learned

Mathematics 31

Course Overview

Mathematics 31 contains 9 units. Beside each unit is a percentage that indicates what the unit is worth in relation to the rest of the course. The units and their percentages are listed below. You will be studying the unit that is shaded.

Unit 1 Introduction to Differential Calculus	10%
Unit 2 Differentiation of Algebraic Expressions and Graphing	10%
Unit 3 Practical Application of Derivatives	20%
Unit 4 Integration	10%
Unit 5 Geometric Vectors and Their Application	**10%**
Unit 6 Algebraic Vectors and Their Application	10%
Unit 7 Inner Product	10%
Unit 8 Systems of Linear Equations	10%
Unit 9 Matrices and Linear Transformations	10%
	100%

Unit Assessment

After completing the unit you will be given a mark based totally on a unit assignment. This assignment will be found in the Assignment Booklet.

Unit Assignment - 100%

If you are working on a CML terminal, your teacher will determine what this assessment will be. It may be

Unit Assignment - 50%
Supervised Unit Test - 50%

Introduction to Geometric Vectors and Their Application

This unit covers topics dealing with geometric vectors and their application. Each topic contains explanations, examples, and activities to assist you in understanding geometric vectors and their application. If you find you are having difficulty with the explanations and the way the material is presented, there is a section called **Extra Help**. If you would like to extend your knowledge of the topic, there is a section called **Extensions**.

You can evaluate your understanding of each topic by working through the activities. Answers are found in the solutions in the **Appendix**. In several cases there is more than one way to do the question.

Unit 5 Geometric Vectors and Their Application

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40%	**Topic 2:** Vector Properties	36
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40%	**Topic 3:** Vector Applications	66
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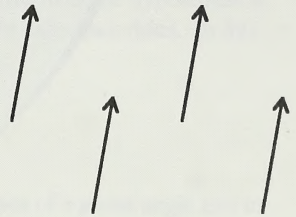
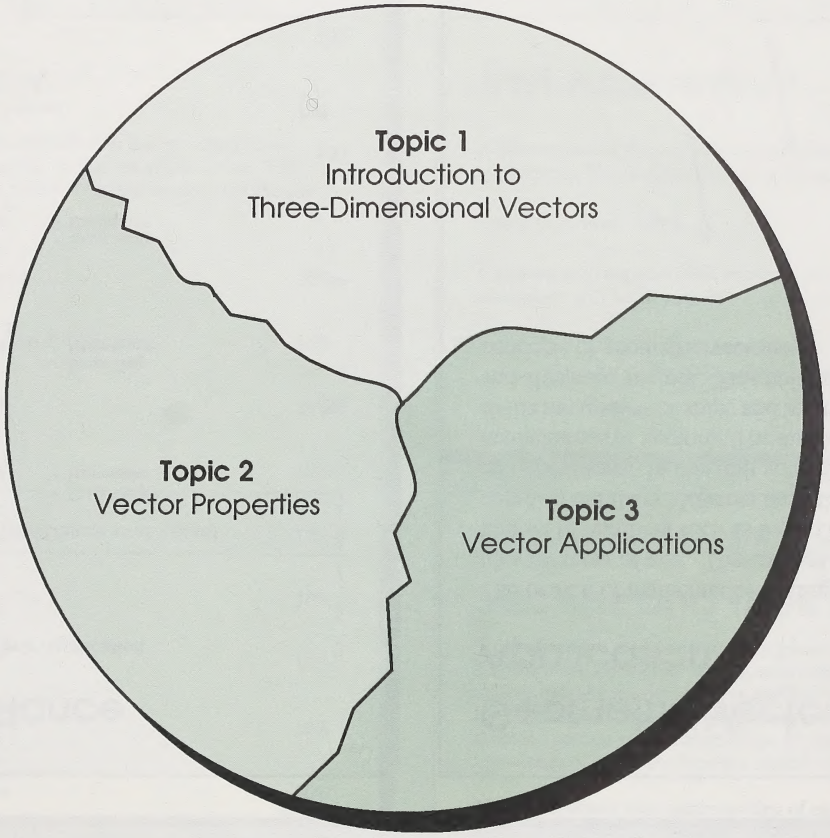
Geometric Vectors and Their Application

The branch of mathematics dealing with vectors is a totally new idea for most of you. The concept of vectors can be compared to a set of numbers such as the set of real numbers or the set of complex numbers. Vectors have their own symbols, notation, and operations. In this unit you will learn the basics about the mathematics of vectors. The application of vectors is important in the navigation of ships and aircraft. There are both geometric and algebraic vectors. This unit will deal only with the concepts of geometric vectors.



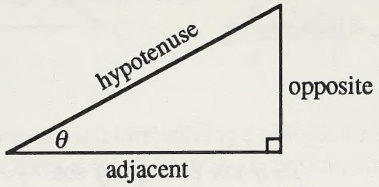
Unit 5 Geometric Vectors and Their Application



What You Already Know

Refresh your memory!

- The following are definitions of the primary trigonometric ratios in right triangles:

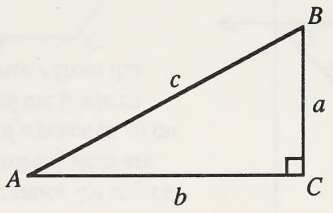


$$\sin\theta = \frac{\text{length of side opposite } \angle\theta}{\text{length of the hypotenuse}}$$

$$\cos\theta = \frac{\text{length of side adjacent } \angle\theta}{\text{length of the hypotenuse}}$$

$$\tan\theta = \frac{\text{length of side opposite } \angle\theta}{\text{length of side adjacent } \angle\theta}$$

- Recall the Pythagorean theorem:



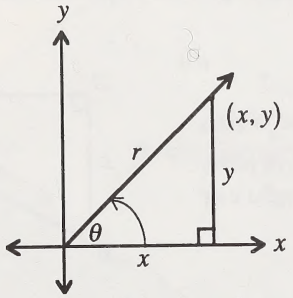
In a right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the other two sides. In this case $c^2 = a^2 + b^2$.

- You should be able to use your calculator to find the primary trigonometric ratios of a given angle and to find an angle given one of the trigonometric ratios.

- You should be able to solve a right triangle using the primary trigonometric ratios.

- You have previously studied trigonometric ratios for obtuse angles. Note that if θ is an obtuse angle – that is, an angle whose measure is greater than 90° but less than 180° – then two of the primary trigonometric ratios, namely the cosine and the tangent, are negative. Why?

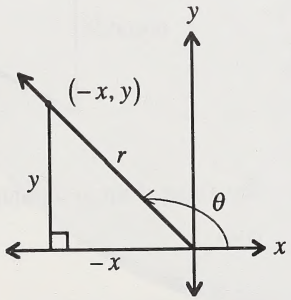
The trigonometric ratios for angles drawn in a coordinate system are defined in terms of a point (x, y) on the terminal arm of the angle in terms of the distance r from this point to the origin.



$$r^2 = x^2 + y^2$$

Therefore, $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, and $\tan\theta = \frac{y}{x}$.

If θ is an obtuse angle, it terminates in the second quadrant and the x-coordinate for any point on the terminal arm is negative.



Therefore, the ratios are as follows:

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{-x}{r} \text{ (which is negative)}$$

$$\tan\theta = \frac{y}{-x} \text{ (which is negative)}$$

Generally, the following statements can be made:

– The sine of an obtuse angle is exactly the same as the sine of its supplement.

$$\sin\theta = \sin(180° - \theta)$$

– The cosine of an obtuse angle has the same numerical value as the cosine of its supplement, but it is negative.

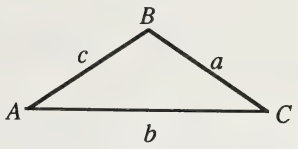
$$\cos\theta = -\cos(180° - \theta)$$

– The tangent of an obtuse angle has the same numerical value as the tangent of its supplement, but it is negative.

$$\tan\theta = -\tan(180° - \theta)$$

- You should be familiar with solving oblique triangles. Oblique triangles do not have a right angle and must be solved using the following:

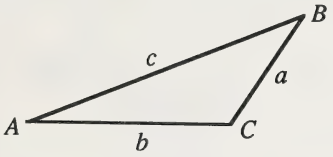
  – The **law of sines** states that the sides of a triangle are proportional to the sines of their opposite angles.



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

  – The **law of cosines** is useful for solving a triangle when two sides and one angle or three sides of the triangle are known.



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Review

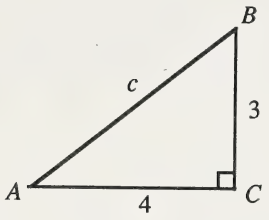
Now try the following review questions.

1. Use ΔABC as shown where $a = 3$ and $b = 4$ to answer the following.

   a. Find c using the Pythagorean theorem.

   b. State the value of the following trigonometric ratios.

   i. $\sin A$ ii. $\sin B$
   iii. $\cos A$ iv. $\cos B$
   v. $\tan A$ vi. $\tan B$



2. In ΔXYZ, $\angle Y = 90°$, $\angle X = 20°$, and $x = 4$. Find y.

3. In ΔDEF, $\angle D = 90°$, $e = 6$, and $d = 10$. Find $\angle F$.

4. Given ΔABC with $\angle C = 90°$, $a = 18.9$, and $\angle A = 40°$, solve the triangle for the unknown sides and angle.

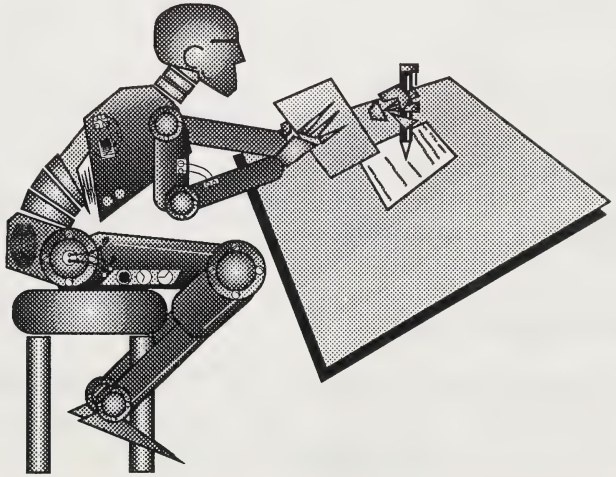
5. Given ΔABC with $\angle C = 100°$, $b = 21$, and $c = 29$, find $\angle B$.

6. Given ΔABC with $\angle A = 50°$, $\angle B = 60°$, and $a = 5$, solve ΔABC.

7. Given ΔABC with $a = 16$, $c = 12$, and $\angle B = 27°$, find b.

8. In ΔABC, $a = 9$, $b = 3$, and $c = 7$. Find the largest angle.

Now go to the **Review** solutions in the **Appendix**.



Topic 1 Introduction to Three-Dimensional Vectors

Introduction

You have worked previously with lines and figures which were drawn in a plane (two-dimensional space). Now you are going to extend these ideas to geometric figures drawn in three-dimensional space.

If you have access to a videocassette recorder (VCR), you may wish to view the video titled **An Introduction to Vectors.** This video provides an interesting and informative introduction to the concepts of geometric vectors. This video is program 12 in the *Catch 31*[1] series.

[1] *Catch 31* is a title of ACCESS Network.

What Lies Ahead

Throughout the topic you will learn to

1. examine three-dimensional geometric relationships
2. examine an introduction to vectors
3. apply vector operations

Now that you know what to expect, turn the page to begin your study of three-dimensional vectors.

Exploring Topic 1

Activity 1

Examine three-dimensional geometric relationships.

In your work in geometry over the last few years, you have studied various figures in a plane. Most of these figures were in two-dimensional space. In this topic you will go one step further. You will be considering geometric figures in three-dimensional space.

Axioms for Three-Dimensional Geometry

First you will consider three axioms for three-dimensional space in order to set the foundation for this topic.

- Two points lying in a plane determine a line which lies entirely in the plane.
- The intersection of two distinct planes is a line.
- Any three distinct noncollinear points form a plane.

As a result of the first axiom, you should understand that a plane and a line **not** lying entirely in the plane can intersect at most in one point. Also, if a line does not intersect the plane nor lie entirely in the plane, then it must be parallel to the plane.

The third axiom is the same for three-dimensional geometry as it is for two-dimensional geometry.

Using a pencil as a line and a sheet of paper as a plane, demonstrate these three situations for yourself. You can demonstrate the second axiom for yourself using two sheets of paper.

Skew Lines

In a plane, lines either intersect or are parallel. There are no other possibilities in euclidean geometry. Here are two planes illustrating intersecting and parallel lines.

Diagram *A*

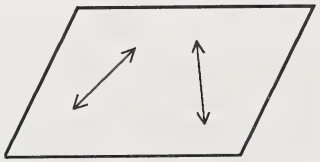
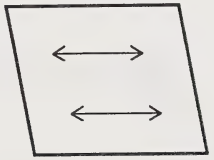


Diagram *B*



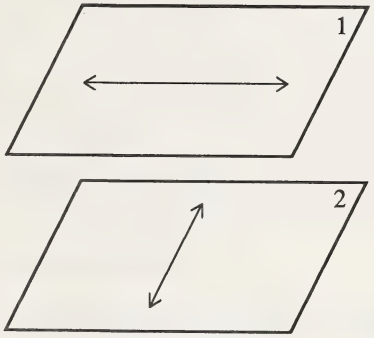
In Diagram *A* the lines will cross at some point if they are extended. In Diagram *B* the lines will never cross even if they are extended to infinity in both directions.

In three-dimensional space (often referred to as three-space) it is possible for two nonparallel lines to not intersect either. Such lines are called **skew lines**.

It is impossible to truly represent skew lines on the plane of a sheet of paper. Rather than just looking at the plane of the page, you must try to visualize depth to the picture.

An easy way to visualize a pair of skew lines is to take two pencils and hold them parallel to each other. Give a slight twist to one of the pencils so that it is out of the plane which was established when the pencils were parallel. You will readily see that two lines will not meet in three-space even if they are nonparallel. (Remember that the pencils each represent a line which extends without bound at each end of the pencil.)

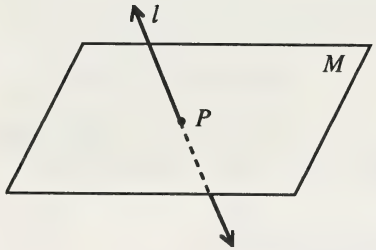
Another way to define skew lines is in terms of whether they are coplanar or not. Two lines are coplanar if they are contained in the same plane. If two lines are noncoplanar, then they are skew.



Here are two planes, one above the other in three-dimensional space.

Intersection of Lines and Planes

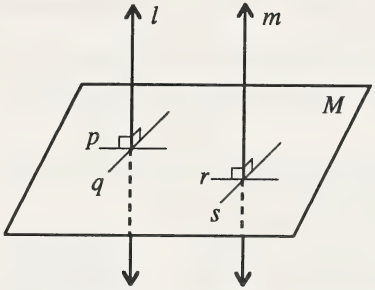
In two-dimensional geometry a line can intersect another line or a line can lie within a plane and be coplanar. In three-dimensional space a line can intersect a plane as shown in the following diagram.



Line l intersects plane M at point P. A dot is used to indicate the point of intersection.

Lines may also intersect a plane in such a way that they are at right angles to the plane.

By definition a line is perpendicular to a plane if it is perpendicular to every line in the plane passing through the point of intersection.

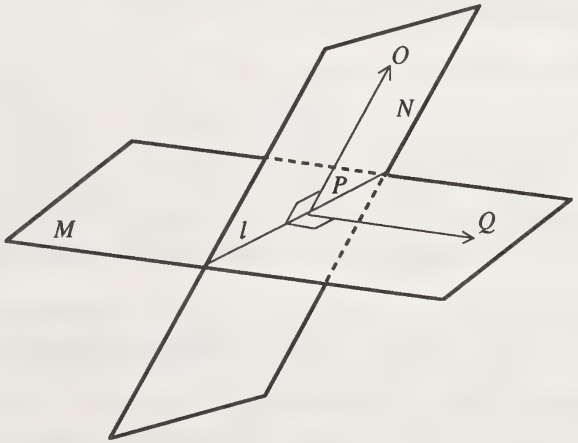


In this diagram line l is perpendicular to lines p and q which are coplanar in plane M. Similarly, line m is perpendicular to plane M.

In plane (two-dimensional) geometry it can be shown that if two lines are perpendicular to the same line, they are parallel to each other. Similarly, in three-space it can be shown that if two lines are perpendicular to a plane, they are parallel to each other.

Dihedral Angles

When two planes intersect, they form an angle called the **dihedral angle**.



In this diagram P is any point on line l which is the intersecting line of the two planes M and N. Ray PO is perpendicular to l, and ray PQ is perpendicular to l. Angle OPQ can be measured in the usual way. In this case it is not a right angle.

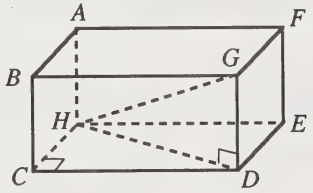
The angle OPQ is called a dihedral angle. By definition the measure of a dihedral angle is the measure of the plane angle formed by the perpendiculars to the line of intersection of the planes at any point on that line.

Problem Solving in Three Dimensions

You should try to follow a simple example that takes you into three-space.

Example 1

The diagram shown represents the rectangular holding tank of a supertanker.



If $CD = 12$ m, $DE = 9$ m, and $BC = 10$ m, calculate the length of the following:

- the length of diagonal GH

Solution:

To find GH, you must first find HD. In ΔHCD, $\angle HCD = 90°$. Therefore, you can apply the Pythagorean theorem. You know $CD = 12$ m and $HC = DE = 9$ m.

$$\begin{aligned} HD^2 &= HC^2 + CD^2 \\ &= (9)^2 + (12)^2 \\ &= 81 + 144 \\ &= 225 \\ HD &= \sqrt{225} \\ &= 15 \end{aligned}$$

In ΔGDH, $\angle GDH = 90°$. You know $GD = BC = 10$ m and you calculated HD to be 15 m.

$$\begin{aligned} GH^2 &= GD^2 + HD^2 \\ &= (10)^2 + (15)^2 \\ &= 100 + 225 \\ &= 325 \\ GH &= \sqrt{325} \\ &\doteq 18 \end{aligned}$$

The length of diagonal GH is 18 m.

- the measure of $\angle DGH$

Solution:

In ΔGDH, GH is the hypotenuse, HD is the side opposite, and GD is the side adjacent. Since you know the measure of all three sides in ΔGDH, you can use any of the three trigonometric functions to calculate $\angle DGH$.

$$\begin{aligned} \sin \angle DGH &= \frac{HD}{GH} \\ &= \frac{15}{18} \\ &= 0.8333 \text{ (to four decimal places)} \\ \therefore \angle DGH &\doteq 56° \text{ (to the nearest degree)} \end{aligned}$$

The measure of $\angle DGH$ is approximately 56°.

Now try the following questions.

1. Sketch the following to show that you understand them.

   a. two planes intersecting in a line

   b. two skew lines

   c. a line and a plane intersecting

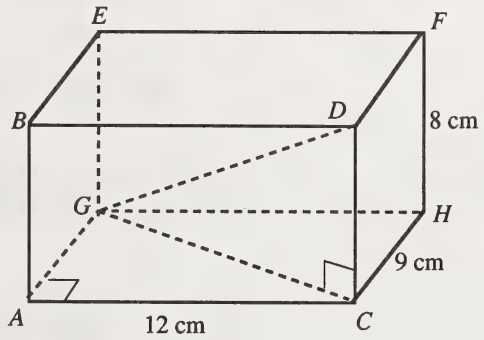
2. How many planes can be made to pass through the following?

   a. a line in space

   b. two intersecting lines in space

   c. two skew lines in space

3. Given the rectangular prism shown in the diagram, answer the questions that follow.

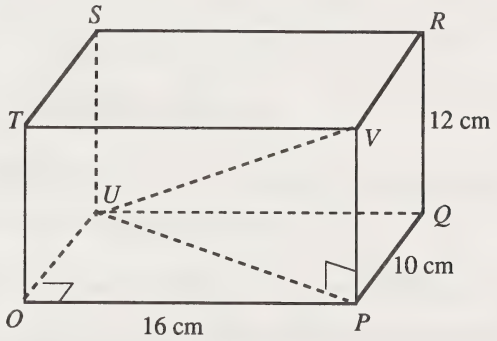


   a. Find the length of diagonal GD.

   b. Find the measure of $\angle DGC$.

4. The angle between the roof of a house and the floor is 30°. If you consider the roof of the house to be contained in one plane and the floor in another plane, what is the measure of the acute dihedral angle between the roof and the wall?

5. The following diagram represents a rectangular trinket box.



   If the side $OU = 10$ cm, $OP = 16$ cm, and $PV = 12$ cm, what are the measures of the following angles? (Round answers to the nearest degree.)

   a. $\angle PUV$

   b. $\angle PUR$

For solutions to **Activity 1**, turn to the **Appendix, Topic 1**.

Activity 2

Examine an introduction to vectors.

Certain quantities can be measured in terms of their size. You could express quantity by a single real number and the appropriate unit. The mass of an object is one such quantity; another is the length of a line segment. In other words a real number and the appropriate unit is enough to state the magnitude of these quantities.

When you use only a magnitude to express the size of a quantity, you refer to it as a **scalar quantity**.

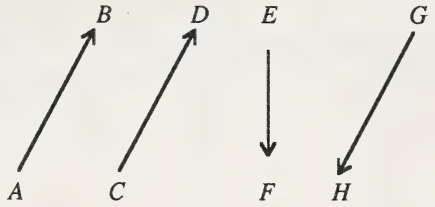
In nature there are many physical quantities which cannot be represented by magnitude alone. For example, a force must be specified by its magnitude as well as the direction in which the force is acting. Such quantities defined by both direction and magnitude are known as **vector quantities**.

Some examples of vector quantities are force, velocity, acceleration, and momentum. Vectors are a fundamental part of mathematics and can be viewed as a mathematical system defined by particular properties with their own particular operations. This is a system in mathematics, just as there is the real-number system and the complex-number system.

A vector is an arrow whose length indicates the magnitude of a vector quantity and whose direction indicates the direction of the vector quantity.

In order to represent a vector, you may use a directed line segment drawn to a specified length. This length is called the magnitude. Such vectors are known as **geometric vectors**.

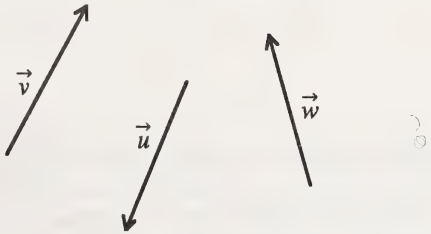
Here are several examples of geometric vectors.



Each vector is drawn with a letter at each of the **initial** and **terminal** points. Vectors *AB*, *CD*, and *GH* are equal in magnitude. Vectors *AB* and *CD* are also equal in direction but vector *GH* is opposite in direction to vectors *AB* and *CD*.

The vectors in the previous diagram may be named $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{EF}$, and $\overrightarrow{GH}$ with the arrow indicating that the direction of $\overrightarrow{AB}$ is from *A* to *B*. The vector $\overrightarrow{BA}$ is not equal to vector $\overrightarrow{AB}$ since they have opposite directions.

Another way to name vectors is with a single letter. For example, the following vectors are named $\vec{v}$, $\vec{u}$, and $\vec{w}$.



This is **not** very useful for geometric representation of vectors since the direction is not explicitly indicated. Use the arrow above the letter to indicate that $\vec{v}$ represents a vector and not a scalar. This notation is useful in algebraic concepts of vectors.

In many applications of vectors, you are interested in the **magnitude** of the vector. The magnitudes of $\overrightarrow{AB}$ and $\vec{v}$ are indicated by $\left|\overrightarrow{AB}\right|$ and $\left|\vec{v}\right|$ respectively.

Referring to the first group of vectors pictured previously, you can write $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{CD}\right| = \left|\overrightarrow{GH}\right|$, since the magnitude of each of these three vectors is the same.

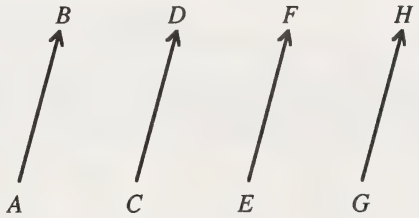
Two vectors are said to be equal if they are equal in magnitude and direction.

The term equivalence is also used when dealing with vectors. You are familiar with equivalent expressions such as $\frac{2}{6}$, $\frac{3}{9}$, $\frac{4}{12}$, and so on.

These ratios all reduce to a common basic value, namely $\frac{1}{3}$.

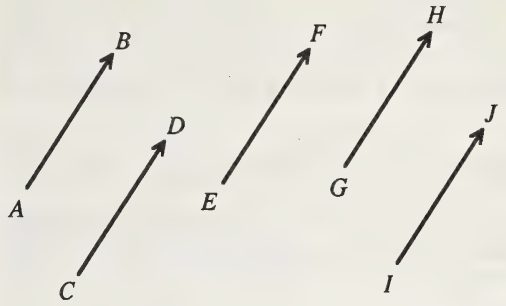
The following are examples of equivalent vectors.

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{GH}$$



When a group of vectors have the same direction and magnitude, they are referred to as an **equivalence class**.

Equivalent vectors are vectors that have the same magnitude and direction.



The preceding vectors are equivalent. You could use $\overrightarrow{AB}$ as a representative of the entire class, since you simply have to slide any one of the other vectors on top of $\overrightarrow{AB}$ to get a match. The equivalent vectors are also called parallel vectors since they all have the same direction.

Before you go on it should be pointed out that a vector may be identified using various notations. Here are some of the more commonly used symbols:

$\overrightarrow{AB}$ $\vec{W}$ v $\vec{u}$

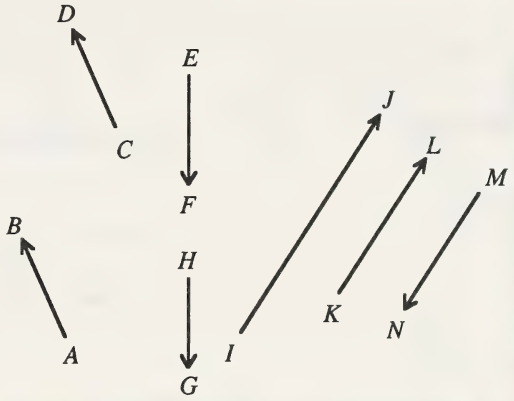
(Note: The boldface letter is not used in this course as a vector symbol.)

Now that you have learned about equivalent vectors, you are ready to take one more step. Consider the aspect of the direction of a vector. If two or more vectors are pointing in the same parallel direction, they may not be headed for the same point, but they are **collinear**.

If they are pointing in the same direction and at the same time are headed for a common point, they are also said to be collinear. Parallel vectors having opposite directions are also collinear. Consider the following definition.

Two or more vectors are said to be collinear if they lie in the same straight line or along a parallel path.

Look at the following vectors.



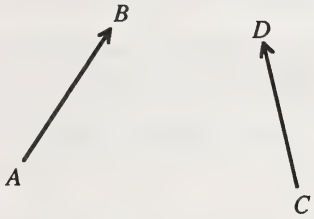
Parallel vectors are collinear since you can **slide** one vector onto the other.

In the diagram $\overrightarrow{AB}$ is collinear with $\overrightarrow{CD}$, $\overrightarrow{EF}$ is collinear with $\overrightarrow{HG}$, and $\overrightarrow{IJ}$ is collinear with $\overrightarrow{KL}$ and $\overrightarrow{MN}$.

It should be pointed out, however, that $\overrightarrow{AB}$ is not collinear with $\overrightarrow{EF}$ nor $\overrightarrow{HG}$. Also, $\overrightarrow{EF}$ is not collinear with $\overrightarrow{IJ}$ nor $\overrightarrow{KL}$ and so on.

Two or more vectors are said to be **coplanar** if they are contained in the same plane.

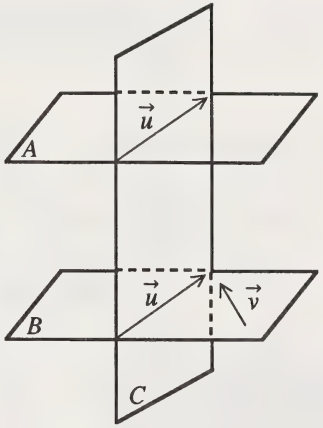
Consider the following pair of vectors.



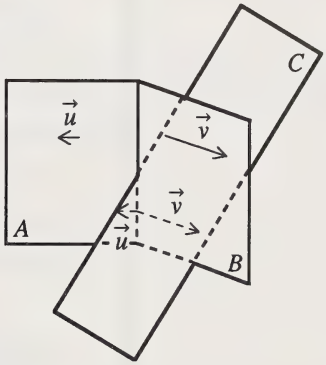
$\overrightarrow{AB}$ and $\overrightarrow{CD}$ are coplanar since they are considered to be contained in the plane of this page.

Two vectors are also coplanar if they are contained in parallel planes.

In the following diagram $\vec{u}$ in plane A and $\vec{v}$ in plane B are coplanar, since you can slide $\vec{u}$ down to plane B.



Now $\vec{u}$ is contained in plane B. Note that two vectors can be coplanar, even if they are contained in nonparallel planes.



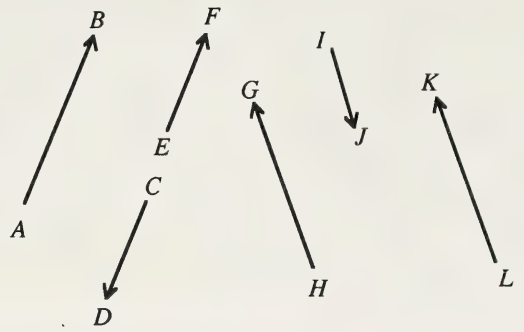
Note that $\vec{u}$ and $\vec{v}$ are coplanar even though planes A and B are nonparallel. If you slide $\vec{u}$ and $\vec{v}$ so that they are tail to tail (the dotted vectors), then the representative vectors of $\vec{u}$ and $\vec{v}$ lie in the same plane. This would be plane C. Therefore, you can say $\vec{u}$ and $\vec{v}$ are coplanar. Try this with a pencil and pen. Let the pencil be $\vec{v}$ and the pen be $\vec{u}$. Hold the pencil and pen in two nonparallel planes A and B as shown in the previous diagram. Now slide the pencil without twisting or turning it until one tip of the pencil touches the tip of the pen. Now try to imagine a plane that includes both the pencil and the pen. This would be plane C as shown in the diagram.

In general you can say that any two vectors are coplanar.

Now try the following questions.

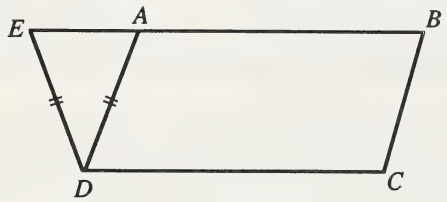
1. Define the following terms.
   a. geometric vectors
   b. equivalent vectors
   c. collinear vectors
   d. coplanar vectors

2. Use these vectors to answer the following.



   a. Name a pair of equivalent vectors.
   b. Name two vectors collinear with $\overrightarrow{EF}$.
   c. Does $\left|\overrightarrow{EF}\right|$ equal $\left|\overrightarrow{IJ}\right|$?
   d. Name one vector having the same magnitude as $\overrightarrow{HG}$.

3. Figure $ABCD$ is a parallelogram and $ED = AD$.



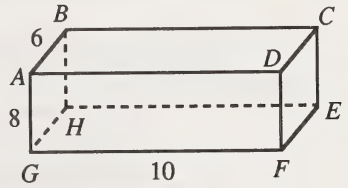
State which vector (if any) is equivalent to the following.

a. $\overrightarrow{AD}$

b. $\overrightarrow{CD}$

c. $\overrightarrow{DE}$

4. Given the rectangular prism, answer the following questions.



a. Name three vectors equivalent to $\overrightarrow{GH}$.

b. Name a vector equivalent to $\overrightarrow{BD}$.

c. Name three vectors collinear with $\overrightarrow{BC}$.

d. Does $|\overrightarrow{HF}|$ equal $|\overrightarrow{DB}|$?

e. Does $|\overrightarrow{GC}|$ equal $|\overrightarrow{BF}|$?

f. Are $\overrightarrow{AD}$ and $\overrightarrow{FE}$ coplanar?

For solutions to **Activity 2**, turn to the **Appendix, Topic 1**.

Activity 3

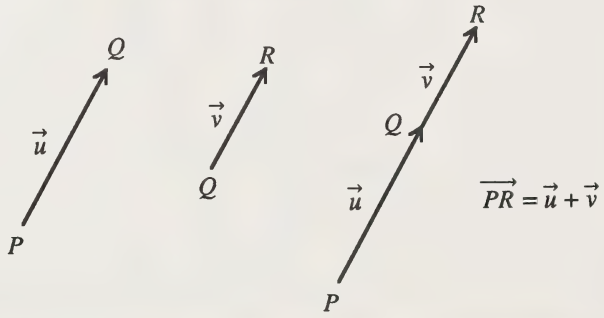
Apply vector operations.

Now that you know what vectors are, you will learn how to perform various operations on them.

First, you will study addition of vectors. When you add two natural numbers such as 6 and 5, what you do is put one number with another number to make a bigger number. For example, $6 + 5 = 11$. You can perform a similar operation with vectors if they are collinear.

Addition of Collinear Vectors

If two vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ have the same direction such as the following, you add them geometrically by placing the tail of the second vector at the head of the first vector.

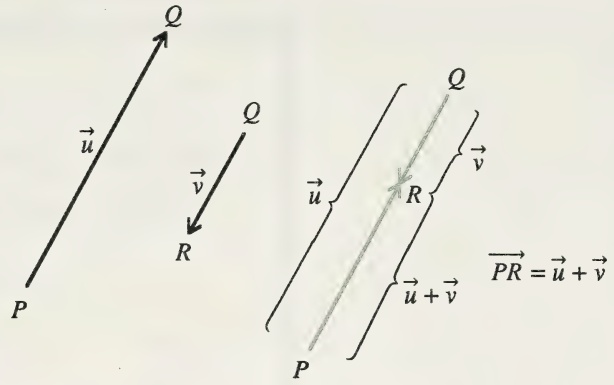


The sum of two or more vectors is a new vector called the **resultant vector**. This new vector extends from the tail of the first vector to the head of the final vector.

The sum of their magnitudes (their absolute values) is the magnitude of the resultant vector (vector sum). This is similar to the addition of real numbers on the number line.

What happens if $\vec{u}$ and $\vec{v}$ are opposite in direction? To add these vectors geometrically, you again place the tail of the second vector on the head of the first vector. The sum of the vectors is a new vector which is from the tail of the first vector to the head of the second vector.

This is very similar to the subtraction of real numbers on the number line. In the following diagram the resultant vector from the addition of $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ is $\overrightarrow{PR}$.



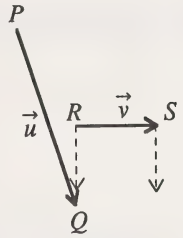
Addition of Noncollinear Vectors

Your next task is to be able to add two noncollinear vectors.

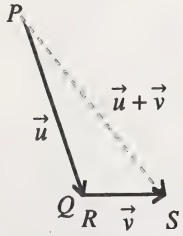
There are two methods of doing this, namely the **triangle method** and the **parallogram method**.

Triangle Method

Suppose you are given two vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ as shown in the following diagram. When you are using the triangle method, you position the vectors as you did with collinear vectors with the initial point of the second vector at the terminal point of the first vector. Be careful not to change the directions or the magnitudes of the vectors.



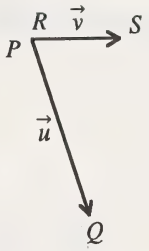
Now complete the triangle.

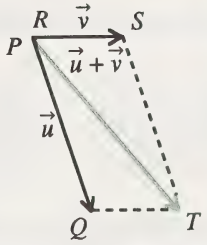


The vector $\overrightarrow{PS}$ is the resultant vector. It is the third side of the triangle. Observe also that $\left|\overrightarrow{PS}\right|$ is less than $\left|\overrightarrow{PQ}\right|+\left|\overrightarrow{RS}\right|$ and $\overrightarrow{PS}$ has a different direction than either $\overrightarrow{PQ}$ or $\overrightarrow{RS}$.

Parallelogram Method

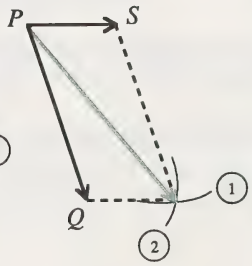
Using the parallelogram method, you position the vectors so that their initial points coincide, that is, so they are at the same point. Then you complete a parallelogram. The resultant is the diagonal of the parallelogram, namely $\overrightarrow{PT}$.





To complete the parallelogram accurately, use a compass and straightedge.

First set your compass to the length of $\overrightarrow{PQ}$. Then place your compass with one point at S and make an arc below $\overrightarrow{PS}$ as shown. (1)
Now set your compass to the length of $\overrightarrow{PS}$. Place your compass with one end at Q and make another arc cutting your first arc. (2)

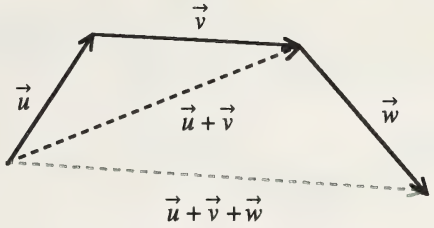


This point of intersection of the two arcs is the fourth corner of the parallelogram. Now you can accurately draw the parallelogram as well as the resultant vector.

To find the resultant (the sum) of two vectors, think of walking along the first vector in the direction indicated, then follow the second vector (after it is placed with its initial point at the terminal point of the first vector). Now think if you started at P and ended up at S, the result is that you went from P to S.

Now that you are able to perform the addition of two vectors, you can now add three vectors, four vectors, and so on by simply adding one more vector to the last resultant.

Here you are given three vectors, namely $\vec{u}$, $\vec{v}$, and $\vec{w}$.

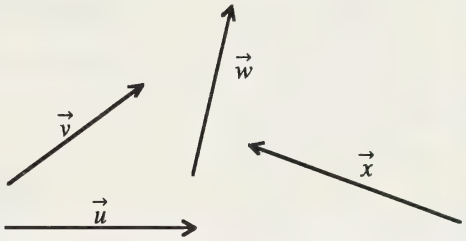


You first add the two vectors $\vec{u}$ and $\vec{v}$. This is equal to $\vec{u}+\vec{v}$ which is a new vector since it has its own magnitude and direction. You now add $\vec{w}$ to this new vector to get another vector $\vec{u}+\vec{v}+\vec{w}$ which is the sum or resultant of vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$. This will continue in the same pattern if you had four vectors, five vectors, and so on.

Now look at a couple examples that deal with the addition of noncollinear vectors.

Example 2

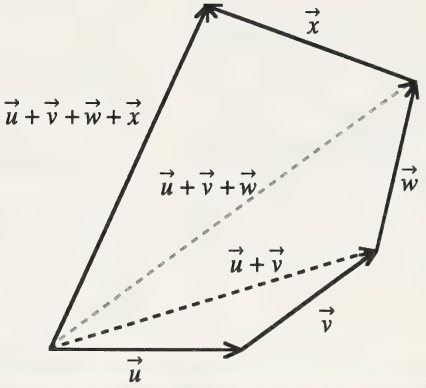
Draw a diagram illustrating the addition of the following vectors, namely $\vec{u}$, $\vec{v}$, $\vec{w}$, and $\vec{x}$.



Solution:

You can combine these vectors using the triangle method.

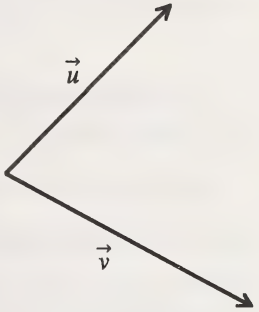
Each vector is placed tail-to-head with the previous one as shown in the following diagram.



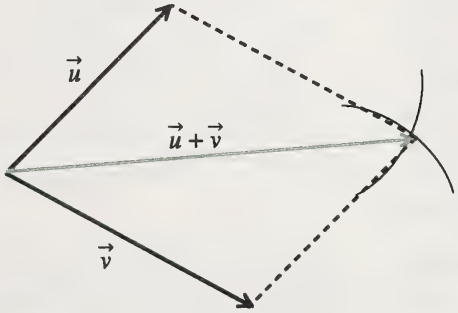
Walking along the vectors $\vec{u}$, $\vec{v}$, $\vec{w}$, and $\vec{x}$ results (overall) in starting at the initial point of $\vec{u}$ and ending up at the terminal point of $\vec{x}$; thus, the resulting vector is named $\vec{u}+\vec{v}+\vec{w}+\vec{x}$ as shown in the diagram.

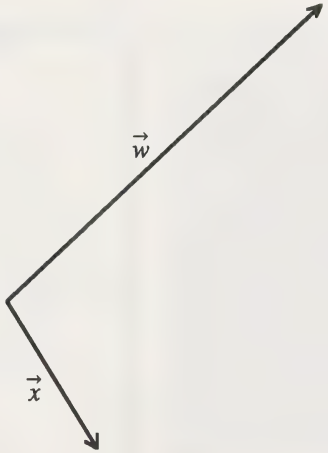
Example 3

Use a compass and a ruler to find the resultant of the following vectors. Use the parallelogram method.

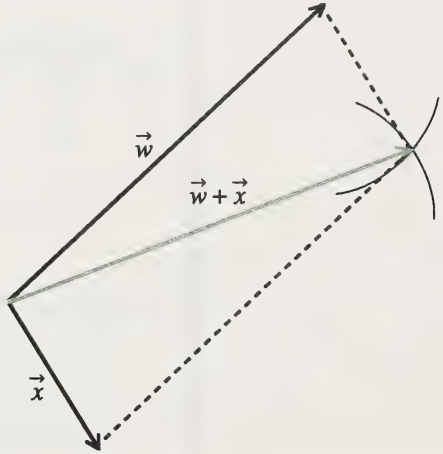


Solution:





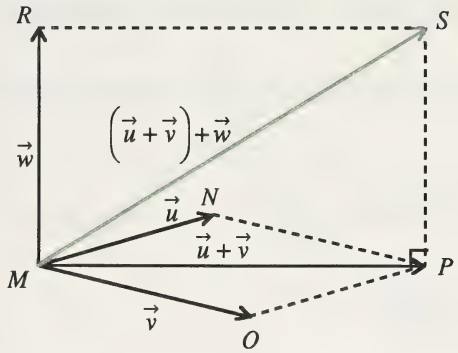
Solution:



The resultant vector gives both the magnitude and the direction of the effect of the two given vectors.

So far you have been confined to vectors in a plane. The rules also apply to vectors in three-space. If you are given $\vec{u}$ and $\vec{v}$ in one plane and $\vec{w}$ in a plane perpendicular to the first, you can combine these vectors to obtain a resultant vector.

Consider the following illustration.



Using the parallelogram method, you first combine $\overrightarrow{MN}$ and $\overrightarrow{MO}$ to obtain $\overrightarrow{MP}$. Then you combine $\overrightarrow{MP}$ with $\overrightarrow{MR}$ to get $\overrightarrow{MS}$. $\overrightarrow{MS}$ is the resultant vector of the three vectors in two different planes.

You will now turn your attention to the inverse of a vector. By definition the negative or inverse of $\vec{u}$ is a vector equal in magnitude but opposite in direction to $\vec{u}$. This inverse is denoted by $-\vec{u}$.

Using geometric notation, the inverse of $\overrightarrow{AB}$ is $-\overrightarrow{AB}$ or $\overrightarrow{BA}$.

What happens when you add a vector to its inverse?

$$\overrightarrow{AB}+\left(-\overrightarrow{AB}\right)$$

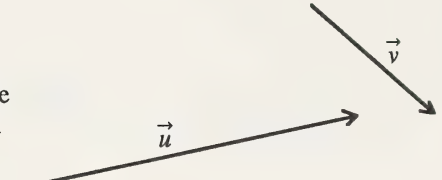
$$=\overrightarrow{AB}+\overrightarrow{BA}$$

$$=\overrightarrow{AA}$$

$$=\vec{0}$$ (This is called the **zero vector**.)

The zero vector, $\vec{0}$, has a magnitude of zero and is considered to have no direction. However, it is convenient to consider the zero vector as parallel to every vector.

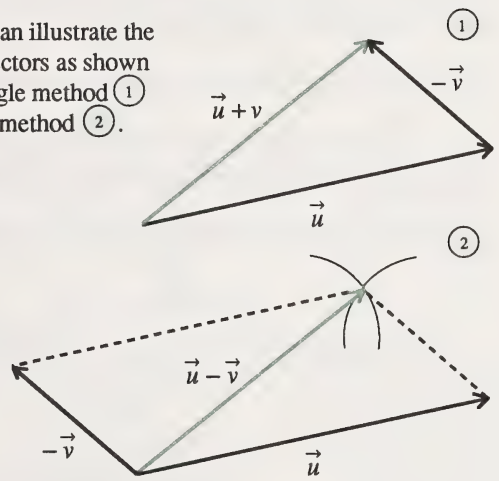
The subtraction of two vectors is analogous to the subtraction of real numbers. If $\vec{u}$ and $\vec{v}$ are two vectors (as shown in this diagram) using the definition of the inverse of a vector, you can express subtraction of vectors as follows:



$$\vec{u}-\vec{v}=\vec{u}+\left(-\vec{v}\right)$$

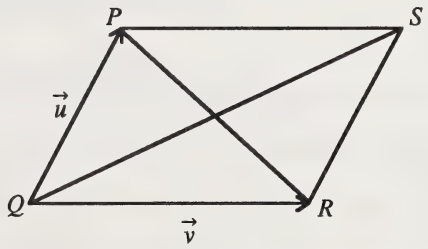
To subtract $\vec{v}$ from $\vec{u}$, you can add the inverse or negative of $\vec{v}$ to $\vec{u}$.

Geometrically, you can illustrate the subtraction of two vectors as shown using either the triangle method (1) or the parallelogram method (2).



In the parallelogram method vectors are placed with the same initial point.

Consider the parallelogram $PQRS$ where $\overrightarrow{QP}=\vec{u}$ and $\overrightarrow{QR}=\vec{v}$.



Here you will see a calculation showing that the shorter diagonal of a parallelogram represents the difference of two vectors while the longer diagonal represents the sum of two vectors.

If $\overrightarrow{QP}=\vec{u}$ and $\overrightarrow{QR}=\vec{v}$, then the following may be stated:

- $\vec{u}+\vec{v}=\overrightarrow{QP}+\overrightarrow{QR}$

  $=\overrightarrow{QP}+\overrightarrow{PS}$ (Opposite sides of a parallelogram are parallel.)

  $=\overrightarrow{QS}$

- $\vec{u}-\vec{v}=\overrightarrow{QP}-\overrightarrow{QR}$

  $=\overrightarrow{QP}+\left(-\overrightarrow{QR}\right)$

  $=\overrightarrow{RS}+\overrightarrow{SP}$

  $=\overrightarrow{RP}$

- $\vec{v}-\vec{u}=\overrightarrow{QR}-\overrightarrow{QP}$

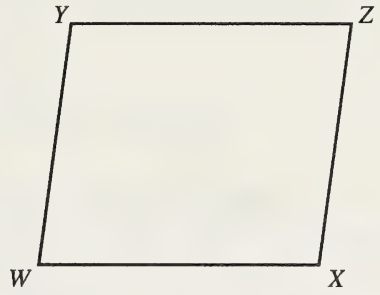
  $=\overrightarrow{QR}+\left(-\overrightarrow{QP}\right)$

  $=\overrightarrow{PS}+\overrightarrow{SR}$

  $=\overrightarrow{PR}$

Now try the following questions. (Do any five.)

1. The diagram shows a parallelogram. State a single vector equal to each of the following.

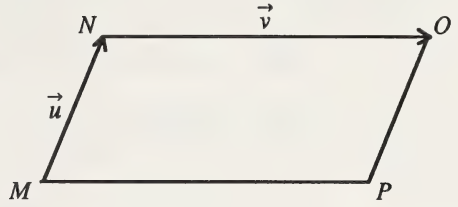


a. $\overrightarrow{WY}$

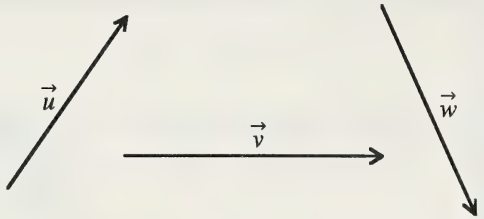
b. $\overrightarrow{YZ}$

c. $\overrightarrow{WX} + \overrightarrow{XZ}$

2. *MNOP* is a parallelogram. State a single vector equal to $\vec{u} + \vec{v}$.



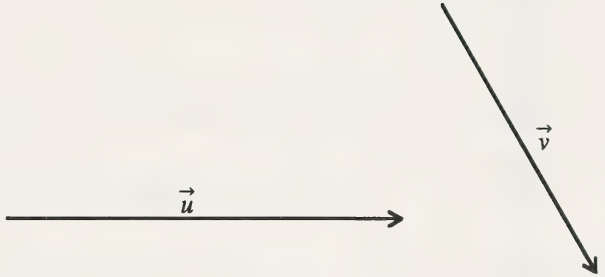
3. Given $\vec{u}$, $\vec{v}$, and $\vec{w}$, find the resultant $\vec{x}$ of these vectors geometrically.



4. Use the given vectors to draw two diagrams by the parallelogram and triangle methods which illustrate the following.

a. $\vec{u} + \vec{v}$

b. $\vec{u} - \vec{v}$

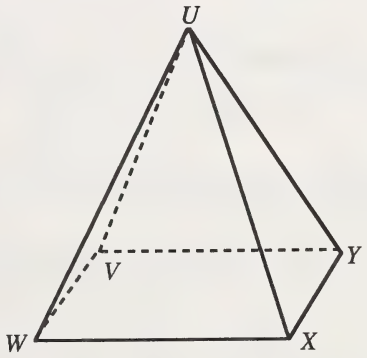


5. The diagram shows a square-based pyramid. State a single vector equal to each of the following.

a. $\overrightarrow{UV} + \overrightarrow{VW}$

b. $\left(\overrightarrow{UY} + \overrightarrow{YX}\right) + \overrightarrow{XW}$

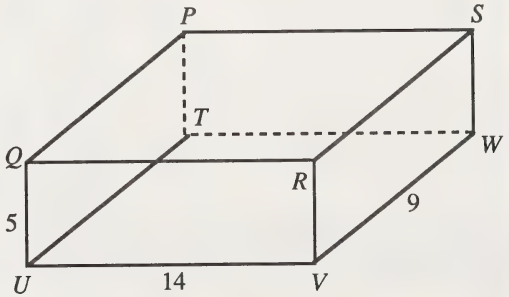
c. $\overrightarrow{UY} + \left(\overrightarrow{YX} + \overrightarrow{XW}\right)$



6. The following diagram is a rectangular prism with $\left|\overrightarrow{UQ}\right| = 5$, $\left|\overrightarrow{UT}\right| = 9$, and $\left|\overrightarrow{UV}\right| = 14$.

a. Calculate $\left|\left(\overrightarrow{UQ} + \overrightarrow{UT}\right) + \overrightarrow{UV}\right|$.

b. Calculate $\angle SUW$ (to the nearest degree).



For solutions to **Activity 3**, turn to the **Appendix, Topic 1**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section. } You may decide to do both.

Extra Help

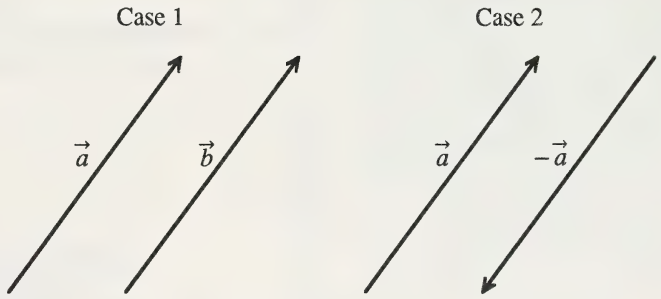
When you speak of a quantity, the first question you might ask is "quantity of what?" The answers to this question could be numerous. Some of these things are mass, displacement, velocity, speed, momentum, force, length, time, acceleration, temperature, or any real number. These quantities can be divided into two categories: quantities that have direction and size; and quantities that have only size. A quantity that has direction and a size (or magnitude) is called a **vector**. A quantity that only has a magnitude is called a **scalar**. From the previous list the vector quantities are velocity, acceleration, displacement, and momentum; the scalar quantities are mass, speed, time, length, temperature, and real numbers.

Vectors may be expressed geometrically as well as algebraically. At this time you will concern yourself with geometric vectors. You will need to know how to perform the various operations on vectors, and you will also study the properties of vectors.

Before you perform vector operations, you should be familiar with these explanations:

- Two vectors $\vec{a}$ and $\vec{b}$ are equal if they have the same magnitude and direction regardless of their starting points.
- A vector having a direction opposite to that of $\vec{a}$ but having the same magnitude is written as $-\vec{a}$.

Consider the following vectors.



In Case 1, $\vec{a}$ and $\vec{b}$ are equal in magnitude and direction.

However, in Case 2, $\vec{a}$ and $-\vec{a}$ are equal in magnitude but opposite in direction.

If you understand these definitions, you can go on to the operation of adding vectors.

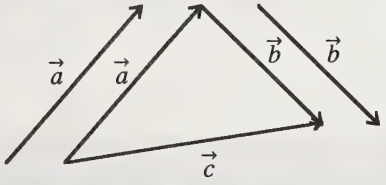
The sum of two or more vectors results in a new vector. This new vector will have its own direction and magnitude. When you take $\vec{a}$ and $\vec{b}$, you can say the sum of $\vec{a}$ and $\vec{b}$ equals some other vector called $\vec{c}$.

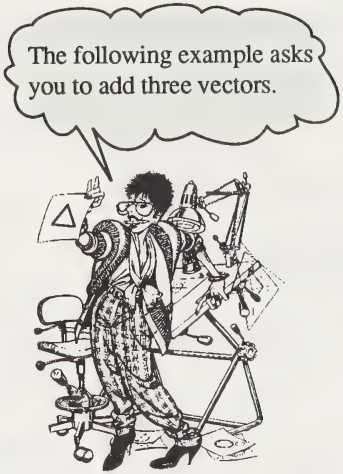
Hence, $\vec{a}+\vec{b}=\vec{c}$. When you add two or more vectors, you get the resultant vector. Therefore $\vec{c}$ is called the resultant of $\vec{a}$ and $\vec{b}$.

It is important to point out that you are not adding real numbers when you do vector addition. The sum is done vectorially.

Look at the following rule.

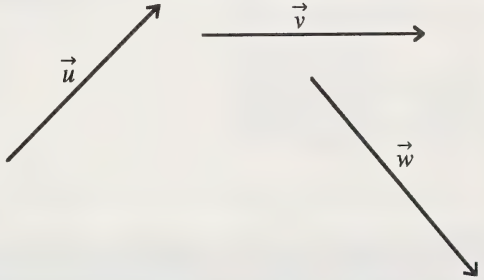
You place the tail of the second vector on the head of the first vector retaining their original directions. You then join the tail of the first vector to the head of the second vector. The line joining these two points is the resultant or the sum of the two original vectors. Study the following figure as you will need it when you do vector addition.

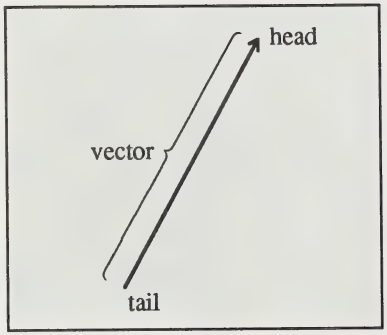




Example 4

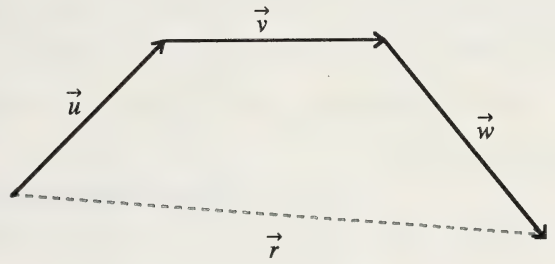
The resultant of $\vec{u}$, $\vec{v}$, and $\vec{w}$ is $\vec{r}$ where $\vec{u}$, $\vec{v}$, and $\vec{w}$ are given in the following diagram. Draw the figure representing the addition of these vectors.





Solution:

Since $\vec{u}+\vec{v}+\vec{w}=\vec{r}$, the figure must retain the directions and the magnitude of the given vectors.



This figure, when complete and closed, is called a vector polygon.

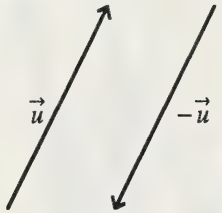
The sum of two vectors can be obtained by completing a triangle of the vectors as well as by finding the greater diagonal of a parallelogram.

When subtracting vectors, the difference of $\vec{a}$ and $\vec{b}$ $\left(\text{represented by } \vec{a}-\vec{b}\right)$ is $\vec{c}$. Then, $\vec{c}$ when added to $\vec{b}$ gives $\vec{a}$; that is, $\vec{b}+\vec{c}=\vec{a}$. It is convenient to write $\vec{a}+\left(-\vec{b}\right)=\vec{c}$.

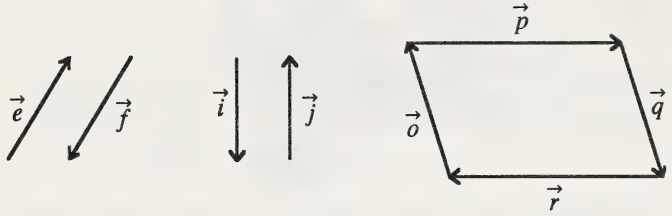
If $\vec{a}=\vec{b}$, then $\vec{a}-\vec{b}=\vec{0}$. Note that $\vec{0}$ is read as the zero vector.

Now try the following questions.

1. What is the relationship between $\vec{u}$ and $-\vec{u}$ in this figure?



2. Study the following figures.



From these vectors indicate all pairs of inverses given one member of each pair.

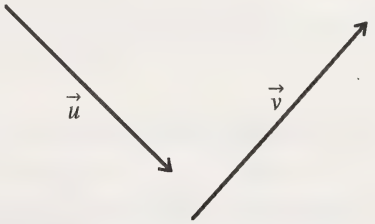
a. $\vec{e}$

b. $\vec{i}$

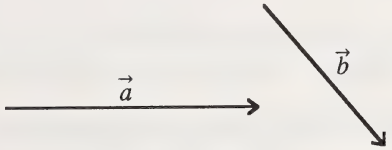
c. $\vec{o}$

d. $\vec{p}$

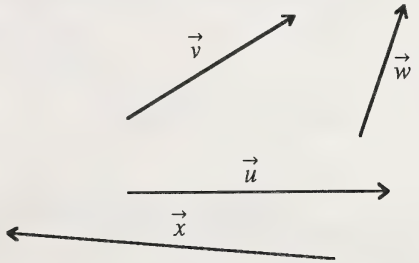
3. Use the triangle method to demonstrate the addition of these two vectors if the resultant is $\vec{w}$.



4. Use the parallelogram method to show the vector difference $\vec{a} - \vec{b}$.



5. Given the vectors $\vec{u}$, $\vec{v}$, $\vec{w}$, and $\vec{x}$ as shown, combine the vectors using the triangle method.



6. Classify each of the following as either a vector or a scalar quantity.

mass	velocity	distance	force
weight	specific heat	momentum	density
calorie	energy	speed	volume
magnetic field	real numbers	displacement	

For solutions to **Extra Help**, turn to the **Appendix, Topic 1**.

Extensions

When forces interact in a given system, the result can be predicted if certain quantities and values are known. Vectors state the magnitude and the direction of forces, and mathematics takes care of the rest.

The following are three examples of vector forces at work.

- You are watching someone trying to row across a river and the current is pulling the boat down the river.
- You imagine that crude oil is flowing in a pipe and friction (viscous drag) is trying to prevent the flow.
- You see a yacht sailing on a lake with the wind and current tugging it in different directions.

Example 5

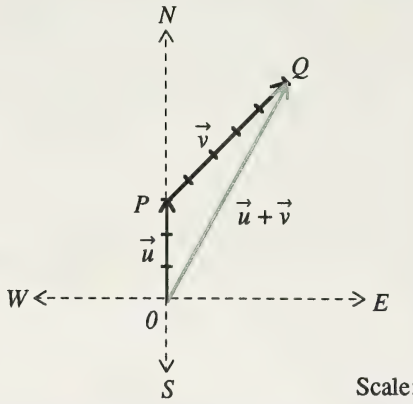
A car travels 3 km due north and then 5 km northeast.

- Use graphical representation to calculate the resultant displacement.

  Solution:

  In order to do this, you construct the cardinal points on your grid. You then measure the units in the respective direction. Put in your vectors and the resultant.

  The resultant force is $\vec{u} + \vec{v}$.



Scale: 1 cm = 2 km

Measuring the resultant on the graph gives a length of approximately 3.7 centimetres. Therefore, the magnitude of the resultant is approximately 7.4 kilometres. The direction of the resultant is approximately 62° north of east.

- Analytically calculate the magnitude and direction of the resultant vector.

  Solution:

  Let $\vec{r} = \vec{u} + \vec{v}$ and use the cosine law.

$$\left|\vec{r}\right|^2 = \left|\vec{u}\right|^2 + \left|\vec{v}\right|^2 - 2\left|\vec{u}\right|\left|\vec{v}\right| \cos \angle OPQ$$

  Since $\overrightarrow{PQ}$ is northeast, $\angle OPQ = 90° + 45° = 135°$.

$$\left|\vec{r}\right|^2 = 3^2 + 5^2 - 2(3)(5)\cos 135°$$

$$= 9 + 25 - 30\left(-\frac{\sqrt{2}}{2}\right)$$

$$= 34 + 15\sqrt{2}$$

$$= 55.2$$

$$\left|\vec{r}\right| \doteq 7.43$$

The resultant displacement is approximately 7.43 km.

To find the direction of the resultant, use the sine law.

$$\frac{\left|\vec{v}\right|}{\sin \angle POQ} = \frac{\left|\vec{r}\right|}{\sin \angle OPQ}$$

$$\frac{5}{\sin \angle POQ} = \frac{7.43}{\sin 135°}$$

$$\sin \angle POQ = \frac{5 \sin 135°}{7.43}$$

$$= \frac{5\left(\frac{\sqrt{2}}{2}\right)}{7.43}$$

$$= 0.4758$$

$$\therefore \angle POQ \doteq 28°$$

The angle you want is $\angle QOE$ which is $90° - 28° = 62°$. Therefore, the direction of the resultant is 62° north of east or 28° east of north.

Notice that you get the same result both graphically and analytically.

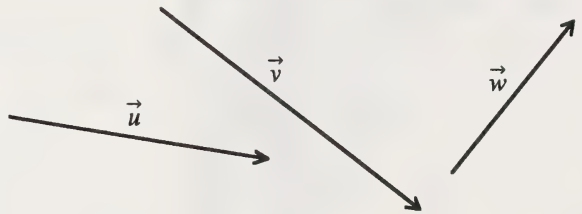
Now try the following questions.

1. A boat sails 4 km due north and 3 km northeast on one of the Great Lakes.

   a. Find the resultant displacement $\vec{r}$ graphically.

   b. Find $\vec{r}$ analytically.

2. Given $\vec{u}$, $\vec{v}$, and $\vec{w}$ as shown, construct a resultant vector equivalent to the following.

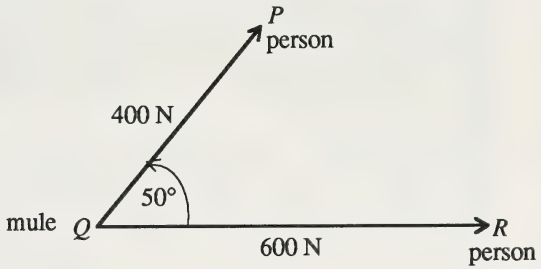


   a. $\vec{v} + 2\vec{w}$

   b. $2\vec{u} - \vec{v}$

   c. $\vec{u} - \vec{v} - 2\vec{w}$

3. Two people are trying to get a stubborn mule onto a truck using two ropes. If the taut ropes are 50° apart and the tensions in the ropes are 400 N and 600 N, what is the resultant force acting on the mule? Give your answer to the nearest newton.



4. Two vectors labelled $\vec{a}$ and $\vec{b}$ have a common initial point with an angle of 40° between the two arms. If

$|\vec{a}| = 14$ and $|\vec{b}| = 10$, find $|\vec{a} + \vec{b}|$ and the angle θ between $(\vec{a} + \vec{b})$ and $\vec{a}$.

For solutions to **Extensions**, turn to the **Appendix, Topic 1**.

Topic 2 Vector Properties

Introduction

Certain properties can be performed with geometric vectors. In this topic you will look at addition and subtraction of vectors using these properties. You will also have discussion on multiplication by a scalar. It is important to note that addition of vectors differs from addition of real numbers. In this topic you will apply vector addition to problems which include definitions of resultant and equilibrant vectors.

What Lies Ahead

Throughout the topic you will learn to

1. define and illustrate vector properties for addition and subtraction

2. define and use the properties of multiplication by a scalar in various problem situations

Now that you know what to expect, turn the page to begin your study of vector properties.

Exploring Topic 2

Activity 1

Define and illustrate vector properties for addition and subtraction.

When you studied real numbers, you learned about certain properties which were applied to them regarding addition. In order to refresh your memory, you will read to recall the following:

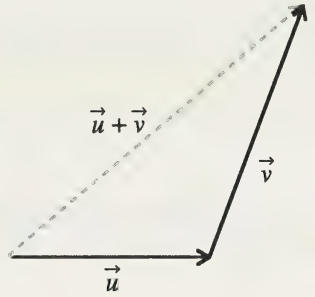
- the **commutative property** which states that $a+b=b+a$
- the **associative property** which states that $(a+b)+c=a+(b+c)$
- the **additive identity property** which states that $a+0=0+a=a$
- the **inverse property** which states that $a+(-a)=(-a)+a=0$

You will now investigate these properties as they apply to vectors.

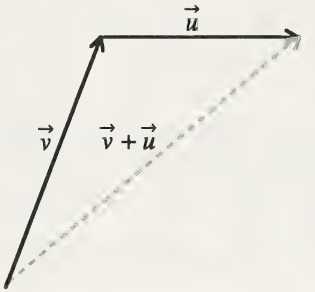
Commutative Property of Vectors

If u and v are vectors, then $\vec{u}+\vec{v}=\vec{v}+\vec{u}$.

In the following two diagrams $\vec{u}$ has the same magnitude and direction. Also, $\vec{v}$ has the same magnitude and direction in both diagrams.



(This illustrates $\vec{u}+\vec{v}$.)



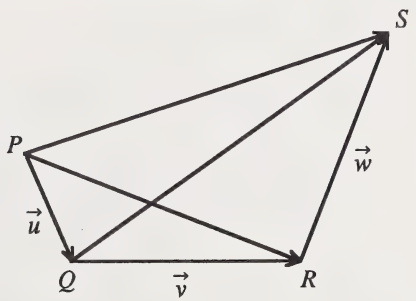
(This illustrates $\vec{v}+\vec{u}$.)

Since $\vec{u}+\vec{v}$ has the same magnitude and direction as $\vec{v}+\vec{u}$, then $\vec{u}+\vec{v}=\vec{v}+\vec{u}$. Hence, the commutative property of addition holds for vectors.

Associative Property of Vectors

Consider such vectors as $\vec{u}$, $\vec{v}$, and $\vec{w}$.
The associative property of vectors states that
$\left(\vec{u}+\vec{v}\right)+\vec{w}=\vec{u}+\left(\vec{v}+\vec{w}\right)$.

Let $\vec{u}=\overrightarrow{PQ}$, $\vec{v}=\overrightarrow{QR}$, and $\vec{w}=\overrightarrow{RS}$ as shown.



$$\left(\overrightarrow{PQ}+\overrightarrow{QR}\right)+\overrightarrow{RS}=\overrightarrow{PR}+\overrightarrow{RS}=\overrightarrow{PS}$$

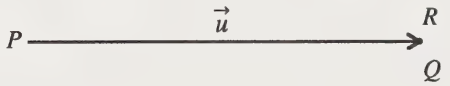
$$\overrightarrow{PQ}+\left(\overrightarrow{QR}+\overrightarrow{RS}\right)=\overrightarrow{PQ}+\overrightarrow{QS}=\overrightarrow{PS}$$

Since the resultant vector is the same, you can state that
$\left(\vec{u}+\vec{v}\right)+\vec{w}=\vec{u}+\left(\vec{v}+\vec{w}\right)$ and the associative property holds for the addition of vectors.

Additive Identity Property of Vectors

Recall that the zero vector $\vec{0}$ has a magnitude of zero and is parallel to every vector. By definition this is the additive identity element for vector addition and is expressed as $\vec{u}+\vec{0}=\vec{0}+\vec{u}=\vec{u}$.

To illustrate this geometrically, if $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are vectors with $\overrightarrow{QR}=\vec{0}$ (that is Q and R coincide), then
$\overrightarrow{PQ}+\overrightarrow{QR}=\overrightarrow{QR}+\overrightarrow{PQ}=\overrightarrow{PQ}$.

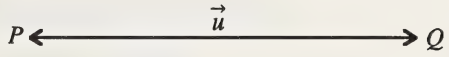


Since the zero vector exists, the additive identity property holds for addition of vectors.

Inverse Property of Vectors

You have previously defined the negative or inverse of a vector $\vec{u}$, denoted by $-\vec{u}$, as a vector equal in magnitude to $\vec{u}$ but opposite in direction. This definition gives lead to the inverse element for the addition of vectors which states that for every vector $\vec{u}$, there is a corresponding vector $-\vec{u}$ such that $\vec{u}+\left(-\vec{u}\right)=\vec{0}$.

This may be stated another way using the following diagram.



$$\overrightarrow{PQ}+\left(-\overrightarrow{PQ}\right)=\overrightarrow{PQ}+\overrightarrow{QP}$$
$$=\overrightarrow{PP}$$
$$=\vec{0}$$

Now study the following example that will touch on all these properties.

Example 1

Prove that $\left(\vec{u}+\vec{v}\right)+\left[\vec{w}+\left(-\vec{u}\right)\right]=\vec{v}+\vec{w}$.

Solution:

$$\left(\vec{u}+\vec{v}\right)+\left[\vec{w}+\left(-\vec{u}\right)\right]=\left(\vec{v}+\vec{u}\right)+\left(-\vec{u}+\vec{w}\right) \quad \text{(by the commutative property)}$$
$$=\left[\left(\vec{v}+\vec{u}\right)+\left(-\vec{u}\right)\right]+\vec{w} \quad \text{(by the associative property)}$$
$$=\left(\vec{v}+\left[\vec{u}+\left(-\vec{u}\right)\right]\right)+\vec{w} \quad \text{(by the associative property)}$$
$$=\left[\vec{v}+\vec{0}\right]+\vec{w} \quad \text{(by the inverse property)}$$
$$=\vec{v}+\vec{w} \quad \text{(by the identity property)}$$

Next you will see the application of the definitions and properties of vectors by looking at some additional examples.

If you want to find the length of a vector or the angle between two vectors, you need to use either the cosine law or the sine law depending on what information you have. Study the following example which illustrates the use of the cosine law and the sine law.

Example 2

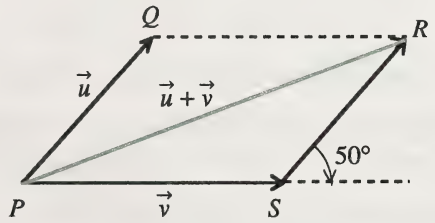
Two vectors $\vec{u}$ and $\vec{v}$ have a common initial point with an angle of 50° between the two arms.

If $|\vec{u}| = 6$ and $|\vec{v}| = 8$, find $|\vec{u} + \vec{v}|$ and the angle between $\vec{u} + \vec{v}$ and $\vec{v}$.

Solution:

Draw a diagram as shown.

To find $|\vec{u} + \vec{v}|$, you must use the law of cosines. Note that $\angle PSR = 130°$ since it is supplementary to the 50° angle.



In ΔPSR, $|\overrightarrow{PS}| = 8$ and $|\overrightarrow{SR}| = 6$.

$$|\overrightarrow{PR}|^2 = |\overrightarrow{PS}|^2 + |\overrightarrow{SR}|^2 - 2|\overrightarrow{PS}||\overrightarrow{SR}| \cos 130°$$

$$= 8^2 + 6^2 - 2(8)(6)\cos 130°$$

$$= 64 + 36 - 96(-0.6428)$$

$$= 64 + 36 + 61.71$$

$$= 161.71$$

$$\therefore |\overrightarrow{PR}| \doteq 12.72$$

Thus, $|\vec{u} + \vec{v}|$ is approximately 12.72 units.

To find the angle between $\vec{u} + \vec{v}$ and $\vec{v}$, namely $\angle RPS$, use the law of sines.

$$\frac{|\overrightarrow{SR}|}{\sin \angle RPS} = \frac{|\overrightarrow{PR}|}{\sin \angle PSR}$$

$$\frac{6}{\sin \angle RPS} = \frac{12.72}{\sin 130°}$$

$$\sin \angle RPS = \frac{(6)(\sin 130°)}{12.72}$$

$$= 0.3613$$

$$\therefore \angle RPS \doteq 21° \text{ (to the nearest degree)}$$

The angle between $\vec{u} + \vec{v}$ and $\vec{v}$ is 21°.

Recall the law of cosines:

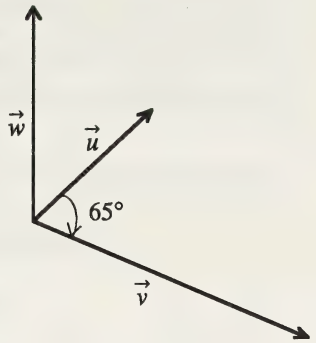
$$c^2 = a^2 + b^2 - 2ab \cos C$$

Recall the law of sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

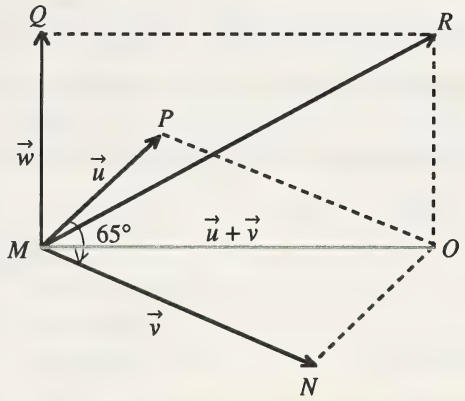
Example 3

Two vectors $\vec{u}$ and $\vec{v}$ have a common initial point and form an angle of 65°. A third vector $\vec{w}$ also has the same initial point and is perpendicular to the plane containing $\vec{u}$ and $\vec{v}$. If $\left|\vec{u}\right| = 8$, $\left|\vec{v}\right| = 14$, and $\left|\vec{w}\right| = 12$,



calculate the magnitude of $\left(\vec{u}+\vec{v}\right)+\vec{w}$ and the angle it makes with the plane containing $\vec{u}$ and $\vec{v}$.

Solution:



Now you must find $\left|\vec{u}+\vec{v}\right|$ or $\left|\overrightarrow{MO}\right|$ in the diagram.

In ΔMNO, $\angle MNO = 180° - 65° = 115°$, $\left|\overrightarrow{MN}\right| = 14$, and $\left|\overrightarrow{NO}\right| = 8$.

Therefore, use the law of cosines.

$$\left|\overrightarrow{MO}\right|^2 = \left|\overrightarrow{MN}\right|^2 + \left|\overrightarrow{NO}\right|^2 - 2\left|\overrightarrow{MN}\right|\left|\overrightarrow{NO}\right| \cos 115°$$

$$= 14^2 + 8^2 - 2(14)(8)(-0.4226)$$

$$= 196 + 64 - 224(-0.4226)$$

$$= 260 + 94.6665$$

$$= 354.6665$$

$$\left|\overrightarrow{MO}\right| \doteq 18.83$$

In ΔMOR, $\angle MOR = 90°$ since $\overrightarrow{MO}$ is in the same plane as $\vec{u}$ and $\vec{v}$, and $\vec{w}$ is perpendicular to this plane.

Therefore, $\left|\overrightarrow{MR}\right|$ can be found using the Pythagorean theorem.

$$\left|\overrightarrow{MR}\right|^2 = \left|\overrightarrow{MO}\right|^2 + \left|\overrightarrow{OR}\right|^2$$

$$= 354.6665 + 144$$

$$= 498.6665$$

$$\left|\overrightarrow{MR}\right| \doteq 22.33$$

Thus, the magnitude of $\left(\vec{u}+\vec{v}\right)+\vec{w}$ is approximately 22.33.

To find the angle between $\overrightarrow{MR}$ and the plane containing $\vec{u}$ and $\vec{v}$, you simply need to find $\angle RMO$ since $\vec{u}$ and $\vec{v}$ are also in this plane.

$$\tan \angle RMO = \frac{\left|\overrightarrow{RO}\right|}{\left|\overrightarrow{MO}\right|}$$

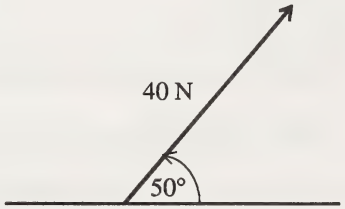
$$= \frac{12}{18.83} = 0.6373$$

$$\therefore \angle RMO \doteq 33° \text{ (to the nearest degree)}$$

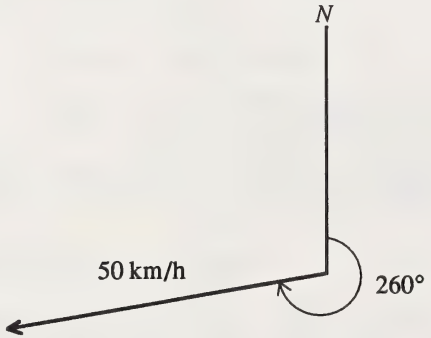
Thus, the angle between $\overrightarrow{MR}$ and the plane containing $\vec{u}$ and $\vec{v}$ is approximately 33°.

Vectors can be applied to many physical problems in which both magnitude and direction are under consideration. In these applications the properties of vectors will be used. The problems considered here will involve force and velocity.

You can represent forces by vectors drawn to indicate magnitude and direction. For example, a force of 40 N acting at an angle of 50° to the horizontal would be represented by a vector drawn at an angle of 50° to the horizontal. If you were to use a scale of 1 cm to represent 10 N, then the length of the vector would be 4 cm as shown in the following diagram.

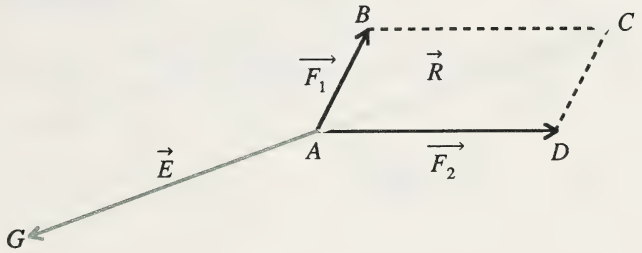


A velocity of 50 km/h at an angle of 260° measured from north can also be represented by a vector of magnitude 50 drawn to some suitable scale.



When two or more forces act on a body, there exists a single force called the **resultant** which has the same effect as the original forces. The resultant of two or more forces acting at a point on a body is that single force which would have the same effect as the given forces. If the physical vectors representing the forces have a common initial point, then the resultant is the **sum** of the given vectors.

In the following diagram you have two forces $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$, represented by $\overrightarrow{AB}$ and $\overrightarrow{AD}$, acting on a body at A.

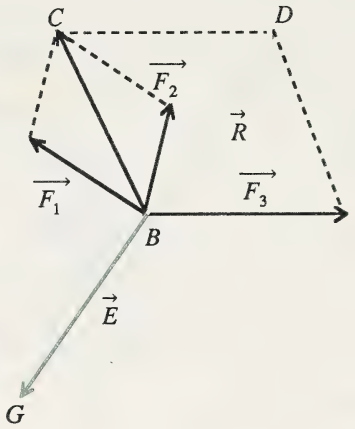


$\overrightarrow{AC}$ represents the magnitude and direction of resultant $\vec{R}$ of $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$. The body would tend to move in the direction of $\overrightarrow{AC}$. However, if an equal force were applied at A in the opposite direction, the body would be maintained in a state of equilibrium.

Such a force, represented by $\overrightarrow{AG}$ in the diagram, is called the **equilibrant**.

The equilibrant of two or more forces acting at a point on a body is that single force which when applied to the body, maintains it in a state of equilibrium.

In the diagram you have three forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$, and $\overrightarrow{F_3}$ acting on a body at B. These three forces are added to determine the resultant represented by $\overrightarrow{BD}$. The equilibrant would be $\overrightarrow{BG}$ or $-\overrightarrow{BD}$.



Velocity can also be used in vector diagrams with the same principles of vector sums applying.

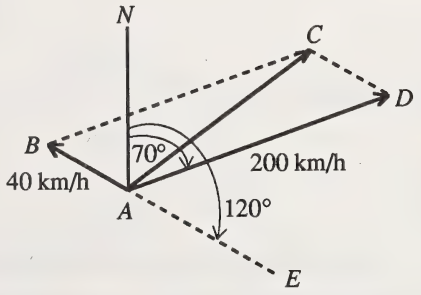
Now look at two more examples.

Example 4

An aircraft is flying at 200 km/h in a direction 70° from north. A wind is blowing at 40 km/h from a direction 120° from north. Determine the actual ground speed and direction of the aircraft.

Solution:

Angles from north are always measured clockwise from north. Therefore, in a direction 70° from north means the aircraft is on a course that is 70° clockwise from north. From a direction 120° from north means the wind is blowing from that direction.



In ΔACD, $\angle CDA = 50°$ since $\angle CDA = \angle DAE$ (alternate angles).

$$\left|\overrightarrow{AC}\right|^2 = \left|\overrightarrow{AD}\right|^2 + \left|\overrightarrow{DC}\right|^2 - 2\left|\overrightarrow{AD}\right|\left|\overrightarrow{DC}\right| \cos \angle CDA$$

$$= (200)^2 + (40)^2 - 2(200)(40) \cos 50°$$

$$= 40\,000 + 1600 - 16\,000(0.6428)$$

$$= 41\,600 - 10\,284.6$$

$$= 31\,315.4$$

$$\left|\overrightarrow{AC}\right| \doteq 177$$

$$\frac{\left|\overrightarrow{DC}\right|}{\sin \angle CAD} = \frac{\left|\overrightarrow{AC}\right|}{\sin \angle CDA}$$

$$\sin \angle CAD = \frac{40 \sin 50°}{177}$$

$$= \frac{40(0.7660)}{177}$$

$$= 0.1731$$

$$\angle CAD \doteq 10°$$

$$\angle CAN = 70° - \angle CAD$$

$$\doteq 60°$$

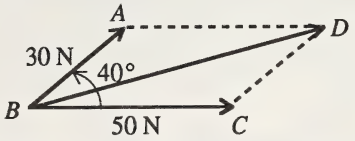
Therefore, the ground speed is 177 km/h in a direction 60° from north.

Example 5

Two forces of 30 N and 50 N act on a body at an angle of 40° to each other. Find the magnitude and direction of the resultant to the nearest newton and nearest degree. Find the magnitude and direction of the equilibrant.

Solution:

First draw a diagram. Since the initial point of each force is the same, the resultant is the diagonal of the completed parallelogram.



$$\angle BCD = 180° - 40°$$
$$= 140°$$

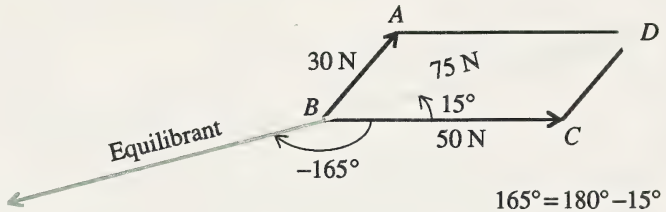
From ΔBCD the following may be stated using the cosine law:

$$\left|\overrightarrow{BD}\right|^2 = \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{CD}\right|^2 - 2\left|\overrightarrow{BC}\right|\left|\overrightarrow{CD}\right| \cos \angle BCD$$
$$= 50^2 + 30^2 - 2(50)(30)\cos 140°$$
$$= 2500 + 900 - (3000)(-0.7660)$$
$$= 3400 + 2298.13$$
$$= 5698.13$$
$$\left|\overrightarrow{BD}\right| \doteq 75$$

The magnitude of the force is approximately 75 N.

To find the direction of the resultant, use the sine law.

$$\frac{\left|\overrightarrow{CD}\right|}{\sin \angle DBC} = \frac{\left|\overrightarrow{BD}\right|}{\sin \angle DCB}$$
$$\sin \angle DBC = \frac{\left|\overrightarrow{CD}\right| \sin \angle DCB}{\left|\overrightarrow{BD}\right|}$$
$$= \frac{30 \sin 140°}{75}$$
$$= \frac{30(0.6428)}{75}$$
$$= 0.2571$$
$$\therefore \angle DBC \doteq 15°$$



$$165° = 180° - 15°$$

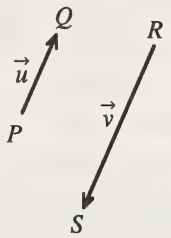
The resultant acts at an angle of 15° to the 50 N force.

The equilibrant would have a magnitude of 75 N and should be applied in the direction −165° with the 50 N force.

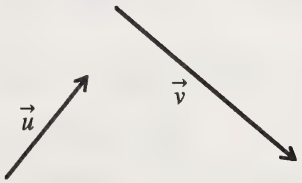
Now try some of the following questions. (Do any five.)

1. Given the following pairs of vectors, draw a diagram to illustrate the sum of the two vectors and indicate the resultant.

a.

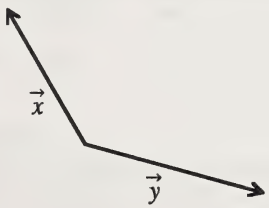


b.

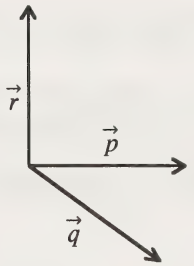


2. For the following vectors use a compass and straightedge to find the resultant of the addition of these vectors.

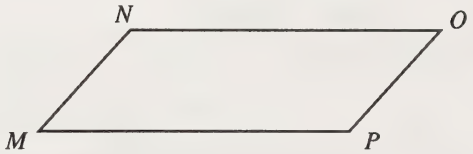
a.



b.



3. *MNOP* is a parallelogram.



State a single vector equal to each of the following.

a. $\overrightarrow{MN} + \overrightarrow{NO}$

b. $\overrightarrow{MP} + \overrightarrow{MN}$

c. $\overrightarrow{OP} + \overrightarrow{PN}$

d. $\overrightarrow{MP} + \overrightarrow{OP}$

e. $\overrightarrow{MN} + \overrightarrow{NO} + \overrightarrow{OP}$

f. $\overrightarrow{PO} + \overrightarrow{OM} + \overrightarrow{MN}$

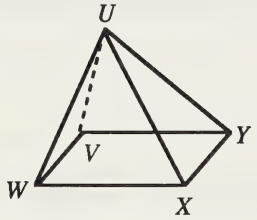
4. The diagram shows a square-based pyramid.

   State single vector equal to each of the following.

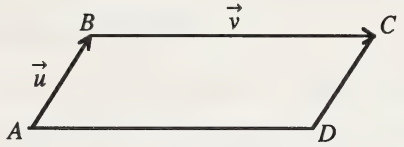
   a. $\overrightarrow{UV}+\overrightarrow{VW}$

   b. $\left(\overrightarrow{UY}+\overrightarrow{YX}\right)+\overrightarrow{XW}$

   c. $\overrightarrow{UY}+\left(\overrightarrow{YX}+\overrightarrow{XW}\right)$



5. *ABCD* is a parallelogram where $\overrightarrow{AB}=\vec{u}$ and $\overrightarrow{BC}=\vec{v}$.



   a. Which vector is equal to $\vec{u}+\vec{v}$?

   b. Which vector is equal to $\vec{v}+\vec{u}$?

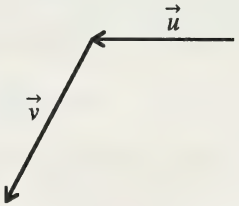
   c. What can be said about $\vec{u}+\vec{v}$ and $\vec{v}+\vec{u}$?

   d. Which vector is equal to $\vec{u}-\vec{v}$?

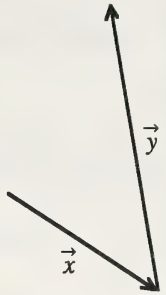
   e. Which vector is equal to $\vec{v}-\vec{u}$?

6. Using a compass and straightedge, construct the resultant of the following vector subtractions.

   a. $\vec{v}-\vec{u}$



   b. $\vec{y}-\vec{x}$



7. Use the vector properties for addition and subtraction to prove the following.

   a. $\left(\vec{u}+\vec{v}\right)+\left(-\vec{u}\right)=\vec{v}$

   b. $\vec{u}+\vec{v}+\left(-\vec{u}\right)+\vec{w}+\left(-\vec{v}\right)=\vec{w}$

8. An aircraft is flying at 450 km/h in a direction 50° from north. A 50 km/h wind is blowing from a direction 90° from north. Calculate the ground speed and direction of the aircraft.

For solutions to **Activity 1**, turn to the **Appendix, Topic 2**.

Activity 2

Define and use the properties of multiplication by a scalar in various problem situations.

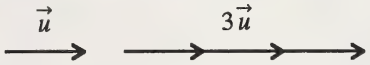
So far you have learned how to add or subtract two vectors. Here you will learn how to multiply a vector by a scalar quantity.

Multiplication by a Scalar

Scalar is just another name for a real number. You call real numbers scalars when they are used in conjunction with vectors. If you have three equal vectors, each with a magnitude and direction represented by $\vec{u}$, then the sum of the three vectors is $\vec{u}+\vec{u}+\vec{u}$. You call this sum $3\vec{u}$ by analogy to the algebra of real numbers.

Thus, $\vec{u}+\vec{u}+\vec{u}=3\vec{u}$ where the real number 3 is called a scalar and $3\vec{u}$ is the multiplication of $\vec{u}$ by the scalar 3.

The meaning of $3\vec{u}$ follows from the above addition. It is a vector in the same direction as $\vec{u}$ but 3 times as long.



On the other hand $-3\vec{u}$ is a vector in the opposite direction of $\vec{u}$ and is three times as long as $\vec{u}$.

Notice that multiplication of a vector by positive scalar other than 1 changes its size but not its direction, whereas multiplication of a vector by a negative scalar other than -1 changes both size and direction. The following defines multiplication by a scalar.

If $\vec{u}$ is a nonzero vector and k is a nonzero real number which is called a scalar, then the vector $k\vec{u}$ can be defined by the following rules:

- $|k\vec{u}|$ and $|k|\vec{u}|$ are equivalent.
- The directions of $k\vec{u}$ and $\vec{u}$ are the same if $k>0$ and opposite if $k<0$.

You can extend this definition to include the zero vector and the scalar zero in the following way:

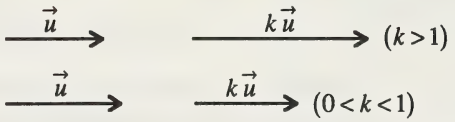
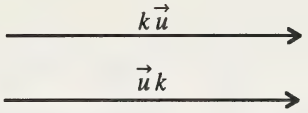
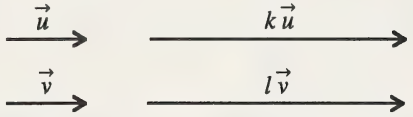
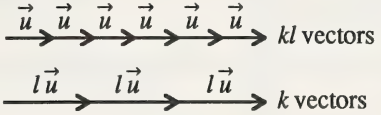
$k\vec{0}=\vec{0}$ (the zero vector) $0\vec{u}=\vec{0}$ (the scalar zero)

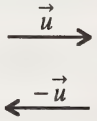
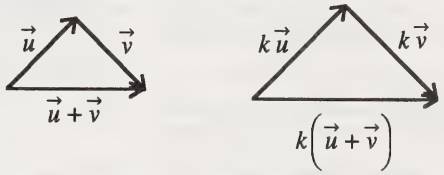
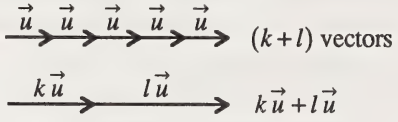
That is, the product of any real number and the zero vector is the zero vector, and the product of zero and any vector is also the zero vector.

Scalar Properties

Earlier, you learned of vector properties as they related to addition. When you discussed scalar quantities, these properties also held. You shall now consider these properties as they relate to multiplication by a scalar.

The following charts list the properties for multiplication by a scalar.

Symbols	Name of Property	Geometric Representation
$k\vec{u}$ is a vector.	Closure	
$k\vec{u} = \vec{u}k$	Commutative	
If $k = l$ and $\vec{u} = \vec{v}$, then $k\vec{u} = l\vec{u}$.	Replacement	
$(kl)\vec{u} = k\left(l\vec{u}\right)$	Associative	
$1\vec{u} = \vec{u}$	Multiplicative Identity Element	The scalar 1 is the multiplicative identity element.

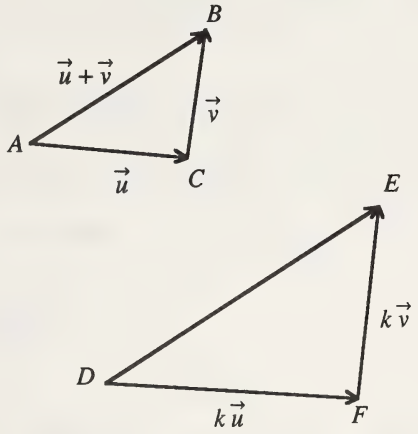
Symbols	Name of Property	Geometric Representation
$(-1)\vec{u} = -\vec{u}$	Additive Inverse	
$k\left(\vec{u}+\vec{u}\right) = k\vec{u} + l\vec{u}$	Distributive (multiplication by a scalar over vector addition)	
$(k+l)\vec{u} = k\vec{u} + l\vec{u}$	Distributive (multiplication by a vector over scalar addition)	
$\left\|k\vec{u}\right\| = \|k\|\left\|\vec{u}\right\|$	The magnitude of a vector $k\vec{u}$ is $\|k\|$ times the magnitude of a vector $\vec{u}$. (The magnitude is expressed in terms of $\|k\|$ in case k is negative.)	

Notice that the distributive property is used twice. Notice also the difference in the geometric representation between the two properties. The validity of these properties may be demonstrated using geometric vectors. Study the following two examples which use geometric vectors to demonstrate the validity of two of the various cases of the distributive property.

Example 6

Show that $k\left(\vec{u}+\vec{v}\right)=k\vec{u}+k\vec{v}$ for $k>0$.

Solution:



Let $\overrightarrow{AC}=\vec{u}$ and $\overrightarrow{CB}=\vec{v}$; then $\overrightarrow{AB}=\vec{u}+\vec{v}$.

Draw ΔEDF with $\overrightarrow{DF}\parallel\overrightarrow{AC}$, $\overrightarrow{FE}\parallel\overrightarrow{CB}$,

$\overrightarrow{DE}\parallel\overrightarrow{AB}$, and $\overrightarrow{DF}=k\vec{u}$.

Then, $\Delta EDF\sim\Delta BAC$.

$$\therefore \overrightarrow{FE}=k\overrightarrow{CB}=k\vec{v}$$

$$\overrightarrow{DE}=k\overrightarrow{AB}=k\left(\vec{u}+\vec{v}\right)$$

$$\overrightarrow{DE}=\overrightarrow{DF}+\overrightarrow{FE}$$

$$=k\vec{u}+k\vec{v}$$

Thus, $k\left(\vec{u}+\vec{v}\right)=k\vec{u}+k\vec{v}$.

Try another example that uses the distributive property. This time there will be two scalar quantities k and l and one vector $\vec{u}$.

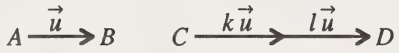
You should pay attention to the restrictions on the quantities – that is k, $l>0$ and $\vec{u}\neq\vec{0}$.

Remember from geometry that $\parallel$ means parallel and $\sim$ means similar.

Example 7

Show that $(k+l)\vec{u} = k\vec{u} + l\vec{u}$ for k, $l > 0$ and $\vec{u} \neq \vec{0}$.

Solution:



Since $k > 0$ and $l > 0$, vectors $k\vec{u}$ and $l\vec{u}$ have the same direction as $\vec{u}$.

$$|k\vec{u}| + |l\vec{u}| = |k||\vec{u}| + |l||\vec{u}|$$

$$= |k + l||\vec{u}| \quad \begin{pmatrix} |k| \text{ and } |l| \\ \text{are scalars} \end{pmatrix}$$

$k\vec{u} + l\vec{u}$ and $(k+l)\vec{u}$ are vectors equal in magnitude and direction.

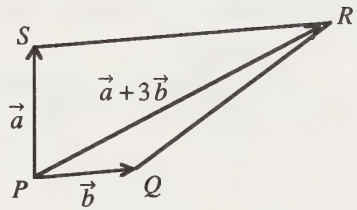
Therefore, $(k+l)\vec{u} = k\vec{u} + l\vec{u}$.

The following are two examples of how the properties can be applied in various ways to manipulate geometric vectors.

Example 8

$PQRS$ is a quadrilateral with $\overrightarrow{PS} = \vec{a}$, $\overrightarrow{PQ} = \vec{b}$, and $\overrightarrow{PR} = \vec{a} + 3\vec{b}$. Express $\overrightarrow{QR}$ and $\overrightarrow{SR}$ in terms of $\vec{a}$ and $\vec{b}$.

Solution:



$$\overrightarrow{QR} = \overrightarrow{PR} - \overrightarrow{PQ}$$

$$= \left(\vec{a} + 3\vec{b}\right) - \vec{b}$$

$$= \vec{a} + 2\vec{b}$$

$$\overrightarrow{SR} = \overrightarrow{PR} - \overrightarrow{PS}$$

$$= \left(\vec{a} + 3\vec{b}\right) - \vec{a}$$

$$= 3\vec{b}$$

$$\overrightarrow{QR} = \overrightarrow{QP} + \overrightarrow{PR}$$

$$= -\overrightarrow{PQ} + \overrightarrow{PR}$$

$$= \overrightarrow{PR} - \overrightarrow{PQ}$$

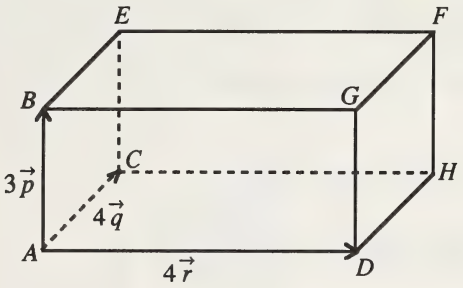
$$\overrightarrow{SR} = \overrightarrow{SP} + \overrightarrow{PR}$$

$$= -\overrightarrow{PS} + \overrightarrow{PR}$$

$$= \overrightarrow{PR} - \overrightarrow{PS}$$

Example 9

$\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$ are three vectors which are 90° to each other as shown in the following diagram. Note that two of the vectors are always in one plane and the third vector is perpendicular to this plane.



- If $\overrightarrow{AB} = 3\vec{p}$, $\overrightarrow{AC} = 4\vec{q}$, and $\overrightarrow{AD} = 5\vec{r}$, express $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}$ in terms of $\vec{p}$, $\vec{q}$, and $\vec{r}$.

  Solution:

$$\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} = 3\vec{p} + 4\vec{q} + 5\vec{r}$$

- If $\left|\vec{p}\right| = 1$, $\left|\vec{q}\right| = 1.5$, and $\left|\vec{r}\right| = 3$, determine $\left|\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}\right|$ in simplest radical form.

  Solution:

$$\left|\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}\right| = \left|\overrightarrow{AF}\right|$$

$$= \sqrt{\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{AD}\right|^2}$$

$$= \sqrt{\left|3\vec{p}\right|^2 + \left|4\vec{q}\right|^2 + \left|5\vec{r}\right|^2}$$

$$= \sqrt{3^2 + 6^2 + 15^2}$$

$$= \sqrt{9 + 36 + 225}$$

$$= \sqrt{270}$$

$$= 3\sqrt{30}$$

A second-order radical is in simplest form when

- the radicand contains no factor that is a perfect square
- the radicand is not a fraction

You will now try a few questions that will give you practice in scalar multiplication. (Do any five.)

1. Use geometric vectors to demonstrate the following.

   a. $3\vec{u}$

   b. $\frac{1}{2}\vec{u}$

   c. $-4\vec{u}$

   d. $2\vec{u}$

2. Show that $k\left(\vec{u}+\vec{v}\right)=k\vec{u}+k\vec{v}$ when $k<0$ and when $k=0$. (Hint: When $k<0$, $k\vec{v}$ is a vector in opposite direction to $\vec{v}$.)

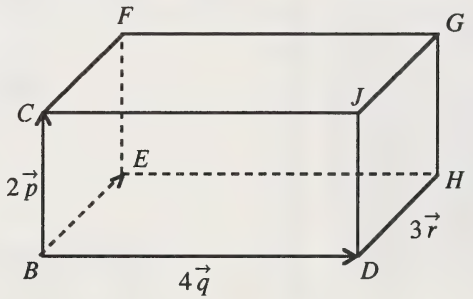
3. $PQRS$ is a quadrilateral where $\overrightarrow{QP}=\vec{a}$, $\overrightarrow{QR}=\vec{b}$, and $\overrightarrow{QS}=2\vec{a}+\vec{b}$. Determine $\overrightarrow{PS}$ and $\overrightarrow{RS}$ in terms of $\vec{a}$ and $\vec{b}$.

4. Quadrilateral $PQRS$ contains $\overrightarrow{QR}=4\vec{a}$, $\overrightarrow{PQ}=2\vec{b}$, and $\overrightarrow{QS}=4\vec{a}-3\vec{b}$. Determine $\overrightarrow{RS}$ and $\overrightarrow{PS}$ in terms of $\vec{a}$ and $\vec{b}$.

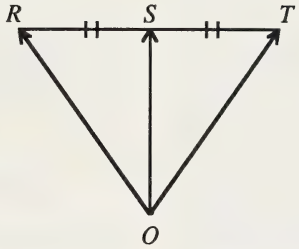
5. In the following diagram $\overrightarrow{BC}$, $\overrightarrow{BD}$, and $\overrightarrow{BE}$ are three vectors at 90° to each other.

   a. If $\overrightarrow{BC}=2\vec{p}$, $\overrightarrow{BD}=4\vec{q}$, and $\overrightarrow{BE}=3\vec{r}$, determine $\overrightarrow{BC}+\overrightarrow{BD}+\overrightarrow{BE}$ in terms of $\vec{p}$, $\vec{q}$, and $\vec{r}$.

   b. If $\left|\vec{p}\right|=1$, $\left|\vec{q}\right|=3$, and $\left|\vec{r}\right|=2$, calculate $\left|\overrightarrow{BC}+\overrightarrow{BD}+\overrightarrow{BE}\right|$.



6. In following diagram O is a point that is not on segment RT. Points R, S, and T are collinear, and S is the midpoint of segment RT. Prove that $\overrightarrow{OR} + \overrightarrow{OT} = 2\overrightarrow{OS}$. (Note that $RS = ST$. Use vector addition or subtraction in each of the triangles.)



For solutions to **Activity 2**, turn to the **Appendix, Topic 2**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section. } You may decide to do both.

Extra Help

Adding and Subtracting Vectors

As you learn more about vectors, you will see that there are many similarities between the real-number system and vectors.

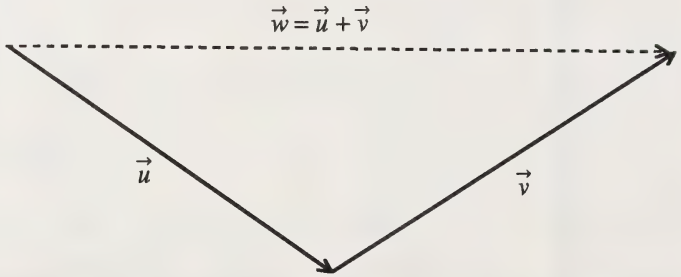
For example, in real numbers if you have $-(-a)$, it is equal to $+a$.

If you have a vector $\vec{u}$, it has a given direction. In order to reverse the direction you add a negative sign. This becomes $-\vec{u}$. If you add another negative sign, you have $-\left(-\vec{u}\right)$ which is equal to $\vec{u}$.

You can add two vectors such as $\vec{u}$ and $\vec{v}$ to form a new vector which you may call $\vec{w}$. You may write this sum as $\vec{w} = \vec{u} + \vec{v}$.

You can even go further by representing this graphically.

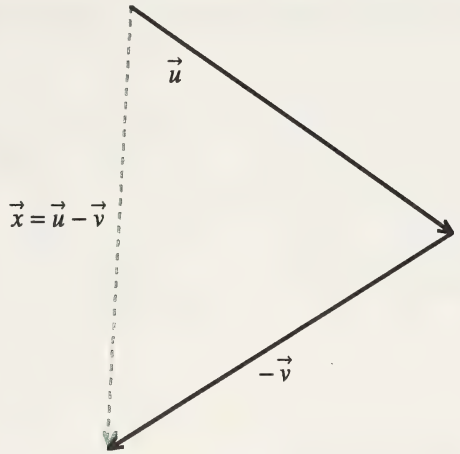
In order to add these two vectors you need to retain their magnitude and direction while placing the tail (initial point) of the second vector on the head (terminal point) of the first vector as shown in the following diagram.



In order to subtract two vectors, you perform an addition using the inverse of the second vector. In other words you add a positive vector to a negative vector. This can be illustrated as $\vec{x} = \vec{u} + \left(-\vec{v}\right)$.

When you represent this graphically, the direction of the second vector $\vec{v}$ is reversed. However, the tail of the second vector still rests on the head of the first vector and the magnitudes of vectors $\vec{u}$ and $\vec{v}$ are still retained.

The following diagram is a representation of the difference of the vectors $\vec{u}$ and $\vec{v}$ which were given previously.



You will notice that in both addition and subtraction of vectors, the geometric shape is a triangle. You can then make the following statements:

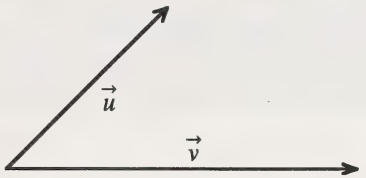
- If $\vec{u}$ and $\vec{v}$ are two vectors and their magnitudes and directions are retained in such a way that the tail of the second rests on the head of the first, then the line joining the tail of the first and the head of the second is the **sum** of the two vectors.

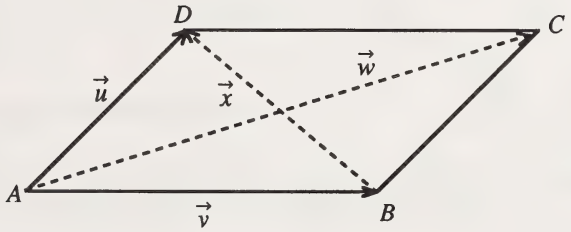
- If $\vec{u}$ and $\vec{v}$ are two vectors and their magnitudes are retained while the direction of the second vector is reversed, then the line joining the tail of the first to the head of the second is the **difference** of the two vectors.

These statments illustrate the triangle method of adding or subtracting vectors.

There is another interesting way of adding and subtracting vectors.

You place the vectors $\vec{u}$ and $\vec{v}$ tail to tail retaining their magnitude and direction. Then complete the parallelogram by drawing segments parallel and equal to $\vec{u}$ and $\vec{v}$ as shown in the following diagrams.





The major diagonal $\overrightarrow{AC}$ is the sum of $\vec{u}$ and $\vec{v}$, or $\vec{w} = \vec{u} + \vec{v}$.

Note that $\vec{w} = \vec{v} + \vec{u}$.

The minor diagonal $\overrightarrow{BD}$ is the difference between $\vec{u}$ and $\vec{v}$, or $\vec{x} = \vec{u} - \vec{v}$. Also note that $\overrightarrow{DB} = \vec{v} - \vec{u}$. This is called the parallelogram method of adding and subtracting vectors.

You will find that if you wish to add or subtract two vectors, you will need to use the cosine law on most occasions. However, when you have enough information you can make use of the sine law. In either case, whether you add or subtract two vectors, the parallelogram method or the triangle method will be necessary.

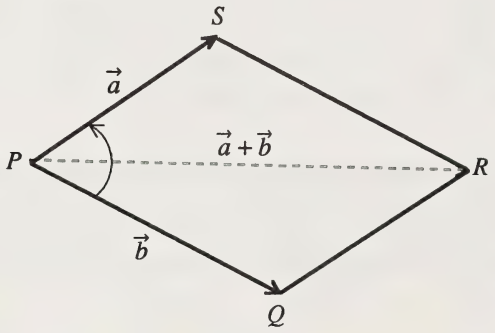
Example 10

Two vectors $\vec{a}$ and $\vec{b}$ have a common initial point and form an angle of 60°. Also, $\left|\vec{a}\right| = 12$ and $\left|\vec{b}\right| = 16$.

- Calculate the value of $\left|\vec{a} + \vec{b}\right|$.

Solution:

First, draw the vectors and complete the parallelogram putting in the major diagonal $\overrightarrow{PR}$. (Note that $\overrightarrow{PR}$ represents the sum of $\vec{a}$ and $\vec{b}$.)



Now solve for $\left|\overrightarrow{PR}\right|$. Since you know only one angle, you cannot use the sine law. Therefore, use the cosine law.

$$\left|\overrightarrow{PR}\right|^2 = \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 - 2\left|\vec{a}\right|\left|\vec{b}\right| \angle PQR$$

$$= 12^2 + 16^2 - 2(12)(16)\cos 120°$$

$$= 144 + 256 - 2(12)(16)(-0.5)$$

$$= 400 + 192$$

$$= 592$$

$$\left|\overrightarrow{PR}\right| = \left|\vec{a} + \vec{b}\right|$$

$$\doteq 24.33$$

- Calculate the angle between the resultant $\left(\vec{a} + \vec{b}\right)$ and $\vec{a}$.

Solution:

The angle between the resultant and $\vec{a}$ is $\angle SPR$. To find this angle you can use the sine law, since you know the size of one angle and the length of its opposite side as well as the length of the side opposite the angle you want to find.

$$\frac{\left|\overrightarrow{RS}\right|}{\sin \angle SPR} = \frac{\left|\overrightarrow{PR}\right|}{\sin \angle PSR}$$

$$\sin \angle SPR = \frac{\left|\overrightarrow{RS}\right| \sin \angle PSR}{\left|\overrightarrow{PR}\right|}$$

$$= \frac{16 \sin 120°}{24.33}$$

$$= \frac{16 \sin 60°}{24.33}$$

$$= \frac{16(0.8660)}{(24.33)}$$

$$= 0.5695$$

$$\therefore \angle SPR \doteq 34.7°$$

Recall: $\cos \theta = -\cos(180 - \theta)$

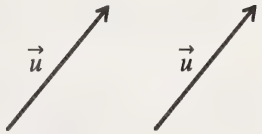
Recall: $\sin \theta = \sin(180 - \theta)$

Multiplication by a Scalar

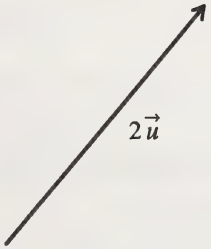
When you speak of multiplication by a scalar, you are really speaking about the product of a vector and a scalar quantity.

You should remember that a real number is a scalar quantity.

Consider two equal vectors, $\vec{u}$ and $\vec{u}$.



If you add them you will get $2\vec{u}$.



Now $\vec{u}$ is a vector and 2 is a real number. When you multiply them, you have the product of a scalar and a vector quantity.

Let k be a scalar quantity and u be some vector $\vec{u}$. If $\vec{u}$ is a nonzero vector and k a nonzero real number, then the vector $k\vec{u}$ is defined as $|k\vec{u}| = |k||\vec{u}|$. Note that $k\vec{u}$ and $\vec{u}$ must have the same direction.

Knowing all this, you can solve a lot of problems relating to vectors.

Try the following example.

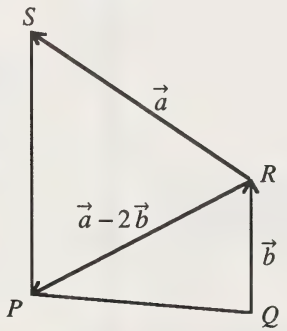
Example 11

$PQRS$ is a quadrilateral where $\overrightarrow{RS} = \vec{a}$, $\overrightarrow{QR} = \vec{b}$, and $\overrightarrow{RP} = \vec{a} - 2\vec{b}$.

- Express $\overrightarrow{QP}$ in terms of $\vec{a}$ and $\vec{b}$.

Solution:

First, draw the figure.

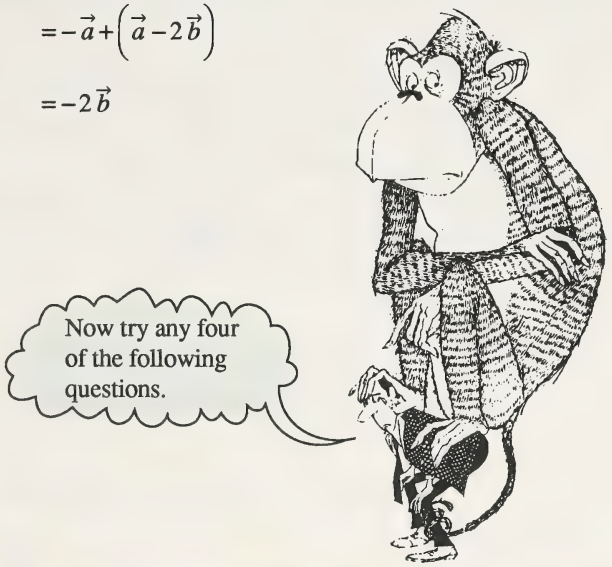


$$\overrightarrow{QP} = \overrightarrow{QR} + \overrightarrow{RP}$$
$$= \vec{b} + \left(\vec{a} - 2\vec{b}\right)$$
$$= \vec{a} - \vec{b}$$

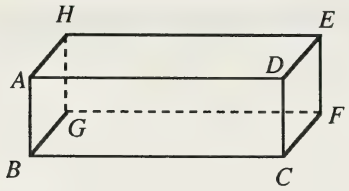
- Express $\overrightarrow{SP}$ in terms of $\vec{a}$ and $\vec{b}$.

Solution:

$$\overrightarrow{SP} = \overrightarrow{SR} + \overrightarrow{RP}$$
$$= -\vec{a} + \left(\vec{a} - 2\vec{b}\right)$$
$$= -2\vec{b}$$



1. Consider the rectangular prism shown. State a single vector equal to each of the following.



a. $\overrightarrow{BA} + \overrightarrow{BC}$

b. $\overrightarrow{BA} + \overrightarrow{BF}$

c. $\left(\overrightarrow{HG} + \overrightarrow{GB}\right) + \overrightarrow{AD}$

d. $\overrightarrow{HG} + \overrightarrow{GC}$

e. $\left(\overrightarrow{BG} + \overrightarrow{BC}\right) + \overrightarrow{FE}$

f. $\overrightarrow{HG} - \overrightarrow{BG}$

g. $\overrightarrow{BC} - \overrightarrow{AB}$

h. $\overrightarrow{HG} - \overrightarrow{DH}$

i. $\overrightarrow{AB} + \overrightarrow{BA}$

j. $\overrightarrow{AB} - \overrightarrow{AB}$

k. $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA}$

l. $\overrightarrow{HG} + \overrightarrow{GD} + \overrightarrow{CG}$

m. $\overrightarrow{BF} - \overrightarrow{EF} + \overrightarrow{EB}$

2. Two equal forces acting at an angle of 90° to each other have a resultant force of 55 N. Calculate the magnitude of the two forces.

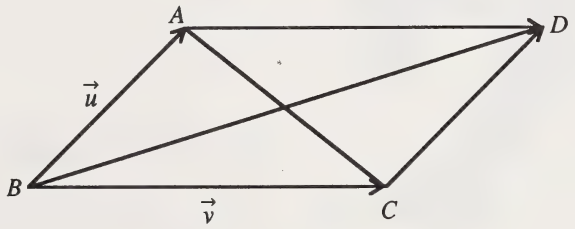
3. Three coplanar forces of x, 20 N, and 30 N act on a body and maintain it in a state of equilibrium. If the x force acts along the horizontal and the 20 N force at an angle of 90° to the horizontal, find the value of x and the direction of the 30 N equilibrant force.

4. Quadrilateral $ABCD$ is drawn such that $\overrightarrow{AD} = \vec{v}$, $\overrightarrow{BA} = 2\vec{u}$, and $\overrightarrow{AC} = 3\vec{v} - 3\vec{u}$.

   Express $\overrightarrow{BC}$ and $\overrightarrow{DC}$ in terms of $\vec{u}$ and $\vec{v}$.

5. In the parallelogram $ABCD$, $\overrightarrow{BA} = \vec{u}$ and $\overrightarrow{BC} = \vec{v}$.

   Show that $\vec{u} + \vec{v} = \overrightarrow{BD}$, $\vec{u} - \vec{v} = \overrightarrow{CA}$, and $\vec{v} - \vec{u} = \overrightarrow{AC}$.



6. An aircraft is flying at 275 km/h in a direction 85° from north. A wind is blowing at 100 km/h from a direction 140° from north. Determine the actual ground speed and direction of the aircraft.

For solutions to **Extra Help**, turn to the **Appendix, Topic 2.**

Extensions

If you would like to do some extra work with this aspect of vectors, the following are a few facts that you should remember.

- $k\left(\vec{u} + \vec{v}\right) = k\vec{u} + k\vec{v}$
- $(k+l)\vec{u} = k\vec{u} + l\vec{u}$
- $\left|k\vec{u}\right| = |k|\left|\vec{u}\right|$
- $\left|\vec{u} + \vec{v}\right| = \sqrt{\left|\vec{u}\right|^2 + \left|\vec{v}\right|^2}$
- The absolute value of a vector quantity is a scalar quantity.

Follow an example using some of these properties.

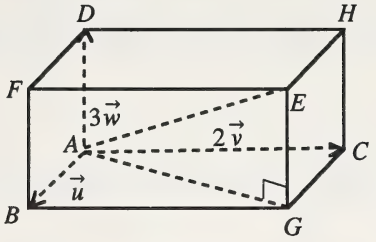
Example 12

$\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$ are three vectors which are 90° to each other. If $\overrightarrow{AB} = \vec{u}$, $\overrightarrow{AC} = 2\vec{v}$, and $\overrightarrow{AD} = 3\vec{w}$, answer the following questions.

- Find $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}$ in terms of $\vec{u}$, $\vec{v}$, and $\vec{w}$.

Solution:

First, draw a vector diagram to represent the given situation.



$$\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} = \vec{u} + 2\vec{v} + 3\vec{w}$$

- If $\left|\vec{u}\right| = 2$, $\left|\vec{v}\right| = 3$, and $\left|\vec{w}\right| = 4$, determine $\left|\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}\right|$.

(Leave your answer in radical form.)

Solution:

$$\left|\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}\right| = \sqrt{\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{AD}\right|^2}$$

$$= \sqrt{\left|\vec{u}\right|^2 + \left|2\vec{v}\right|^2 + \left|3\vec{w}\right|^2}$$

$$= \sqrt{2^2 + [(2)(3)]^2 + [(3)(4)]^2}$$

$$= \sqrt{4 + 36 + 144}$$

$$= \sqrt{184}$$

$$= 2\sqrt{46}$$

Two vectors $\overrightarrow{R_1}$ and $\overrightarrow{R_2}$ are said to be parallel if $\overrightarrow{R_2} = \frac{\overrightarrow{R_1}}{\left|\overrightarrow{R_1}\right|}$ where $\overrightarrow{R_2}$ is called a unit vector.

$\overrightarrow{R_2}$ is parallel to $\overrightarrow{R_1}$ because $\overrightarrow{R_2}$ is a scalar multiple of $\overrightarrow{R_1}$. (A vector of length 1 is a unit vector.)

The following example shows you how to determine a unit vector.

Example 13

Given $\vec{a} = 2\vec{u} + 4\vec{v} - 5\vec{w}$ and $\vec{b} = \vec{u} + 2\vec{v} + 3\vec{w}$ where $\left|\vec{u}\right| = 2$, $\left|\vec{v}\right| = 4$, and $\left|\vec{w}\right| = 5$, find a unit vector that is parallel to the resultant of vectors $\vec{a}$ and $\vec{b}$. Verify that your answer is correct.

Solution:

Let $\overrightarrow{R_1}$ be the resultant vector.

$$\begin{aligned}\overrightarrow{R_1} &= \vec{a} + \vec{b} \\ &= \left(2\vec{u} + 4\vec{v} - 5\vec{w}\right) + \left(\vec{u} + 2\vec{v} + 3\vec{w}\right) \\ &= 3\vec{u} + 6\vec{v} - 2\vec{w}\end{aligned}$$

$$\begin{aligned}\therefore \left|\overrightarrow{R_1}\right| &= \left|3\vec{u} + 6\vec{v} - 2\vec{w}\right| \\ &= \sqrt{\left[3(2)\right]^2 + \left[6(4)\right]^2 - \left[2(5)\right]^2} \\ &= \sqrt{6^2 + 24^2 - 10^2} \\ &= \sqrt{36 + 576 - 100} \\ &= \sqrt{512}\end{aligned}$$

A unit vector $\overrightarrow{R_2}$ parallel to $\overrightarrow{R_1}$ is $\frac{\overrightarrow{R_1}}{\left|\overrightarrow{R_1}\right|}$.

$$\therefore \overrightarrow{R_2} = \frac{\overrightarrow{R_1}}{\left|\overrightarrow{R_1}\right|}$$

$$= \frac{3\vec{u} + 6\vec{v} - 2\vec{w}}{\sqrt{512}}$$

Verification:

$$\left|\frac{3\vec{u}}{\sqrt{512}}+\frac{6\vec{v}}{\sqrt{512}}-\frac{2\vec{w}}{\sqrt{512}}\right|=\sqrt{\left[\frac{3(2)}{\sqrt{512}}\right]^2+\left[\frac{6(4)}{\sqrt{512}}\right]^2-\left[\frac{2(5)}{\sqrt{512}}\right]^2}$$

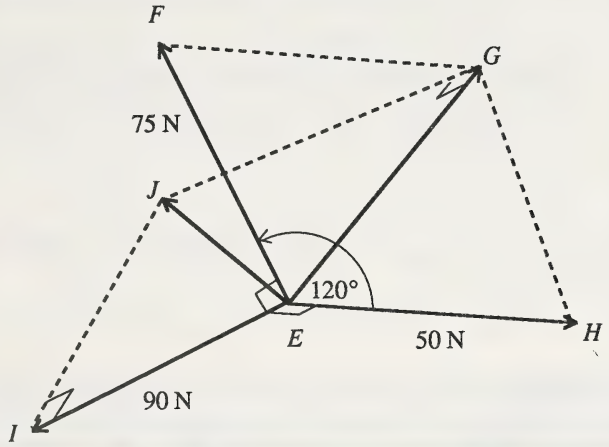
$$=\sqrt{\frac{36+576-100}{512}}$$

$$=\sqrt{\frac{512}{512}}$$

$$=1$$

Now try the following questions.

1. Three forces of 50 N, 75 N, and 90 N act on a body. The first two act at an angle of 120° to each other, and the third is perpendicular to the plane of the first two. Find the magnitude of the resultant (to the nearest newton) and the angle the resultant makes with the plane of the first two forces.



2. Given $\vec{r}=3\vec{u}-\vec{v}+4\vec{w}$ with $\left|\vec{u}\right|=2$, $\left|\vec{v}\right|=3$, and $\left|\vec{w}\right|=1$, find the unit vector $\overrightarrow{R_2}$ which is parallel to $\vec{r}$.

3. Given $\vec{a}=3\vec{u}-2\vec{v}+\vec{w}$, $\vec{b}=2\vec{u}-4\vec{v}-3\vec{w}$, and $\vec{c}=-\vec{u}+2\vec{v}+2\vec{w}$ where $\vec{u}$, $\vec{v}$, and $\vec{w}$ are unit vectors, find the magnitude of the following. (Leave your answers in radical form.)

   a. $\vec{b}$

   b. $\vec{a}+\vec{b}+\vec{c}$

   c. $\vec{a}-2\vec{b}-3\vec{c}$

For solutions to **Extensions**, turn to the **Appendix, Topic 2**.

Topic 3 Vector Applications

Introduction

In this topic you will learn how to resolve a vector into its rectangular components and see how these rectangular components apply in problem situations. Linear combinations of geometric vectors will also be determined.

What Lies Ahead

Throughout the topic you will learn to

1. define and apply resolution of forces
2. define and apply linear combinations of geometric vectors

Now that you know what to expect, turn the page to begin your study of vector applications.

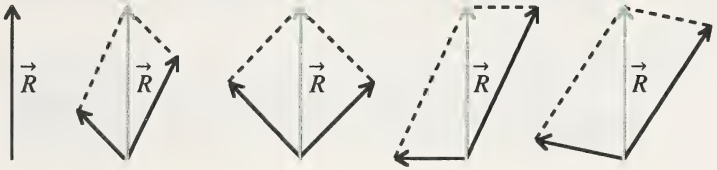
Exploring Topic 3

Activity 1

Define and apply resolution of forces.

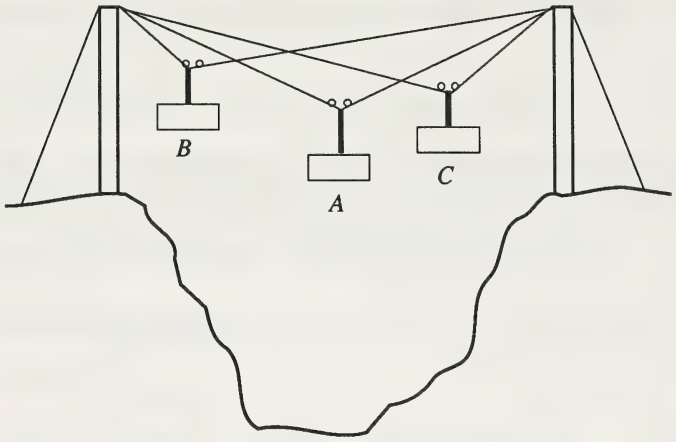
Components of a Force

A single force may be broken into **components**. Previously, you have seen that two vectors can be added to find a resultant $\vec{R}$. The vectors are called the components of $\vec{R}$. Thus, a single force can be resolved into its components. These components can be in any direction. In the following diagram the same resultant force $\vec{R}$ has been resolved into four possible pairs of component forces.



Actually, a force may be resolved into an infinite number of pairs of components.

A dam is being built over a canyon. Consider a load of cement being transported across the canyon, by means of a container suspended on a cable as pictured in the following diagram.



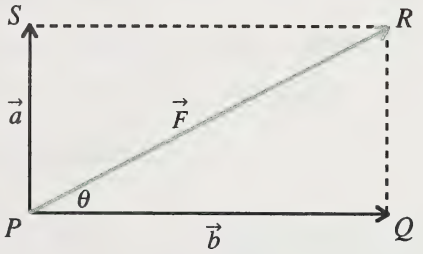
As the container moves along the cable, the force opposing the weight of the container would be resolved into two different pairs of components at different points along the way. If you want to find the tension in the cables, you would be interested in the components along the direction of the cables.

Rectangular Components

While it is possible to resolve a force into non-rectangular components, most useful forces are those which you obtain by resolving the force into rectangular components.

Rectangular components are those components which are **perpendicular** to each other.

In the following diagram a force $\vec{F}$ has been resolved into two rectangular components $\vec{a}$ and $\vec{b}$ so that the angle between $\vec{F}$ and $\vec{b}$ is θ.



Now you can write the following:

$$\frac{\overrightarrow{PQ}}{\overrightarrow{PR}} = \cos\ \theta$$

$$\overrightarrow{PQ} = \overrightarrow{PR}\cos\ \theta$$

$$\therefore \left|\vec{b}\right| = \left|\vec{F}\right|\cos\ \theta$$

$$\frac{\overrightarrow{PS}}{\overrightarrow{PR}} = \cos(90° - \theta)$$

$$\overrightarrow{PS} = \overrightarrow{PR}\cos(90° - \theta)$$

$$= \overrightarrow{PR}\sin\ \theta$$

Therefore, $\left|\vec{a}\right| = \left|\vec{F}\right|\sin\ \theta$.

You can now make the statement that the force $\vec{F}$ has two components – one vertical and one horizontal.

The vertical component $\vec{a}$ is $\left|\vec{F}\right|\sin\ \theta$ and the horizontal component $\vec{b}$ is $\left|\vec{F}\right|\cos\ \theta$.

Problems related to the inclined plane are typical of questions requiring you to resolve forces into rectangular components.

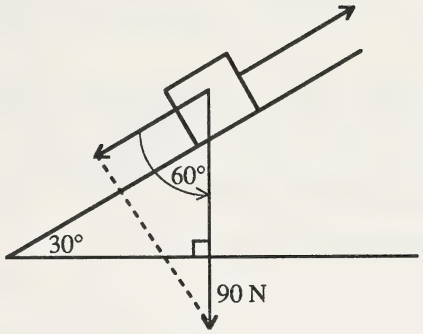
Study the following examples.

Example 1

What force in the direction of the plane is required to maintain a body with a weight of 90 N in equilibrium on a smooth (frictionless) ramp which is at an angle of 30° to the horizontal?

Solution:

First you draw and label a diagram.



In order to maintain the body on the ramp, you need to counteract its tendency to slide down the ramp. This tendency to slide down the ramp is not the entire 90 N weight, but is represented by the component of the 90 N weight which is parallel to the plane.

If you let the required force be $\vec{P}$, then the following may be stated:

$$\left|\vec{P}\right| = 90 \cos 60°$$
$$= 90(0.5)$$
$$= 45$$

Therefore, the force is 45 N in the direction of the ramp.

This example is a case of application of rectangular components restricted to two-dimensional situations. You can, however, extend the application to three dimensions. In the three-dimensional situation you simply resolve the force in each plane separately.

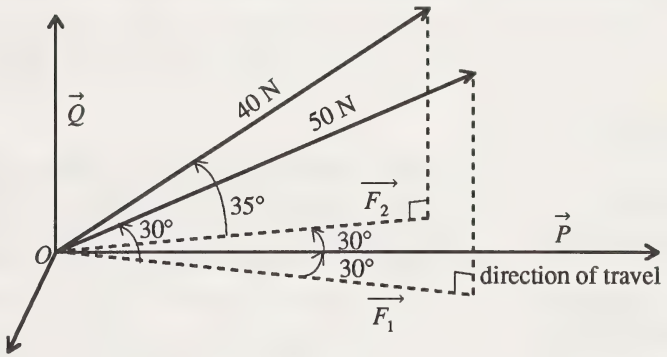
You will now try an example that has forces in three dimensions.

Example 2

Two men are pulling a snowmobile along a smooth surface (frictionless). The first man is pulling with a force of 50 N and at an angle of 30° to the horizontal. The second man is pulling with a force of 40 N and at an angle of 35° to the horizontal. The angle between the ropes is 60° (30° on either side of the path of the snowmobile).

- Find the force that draws the snowmobile forward.

Solution:



Let the force pulling the snowmobile forward be $\vec{P}$.

Let the horizontal component of the applied forces be $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$.

$\therefore \left|\overrightarrow{F_1}\right| = 50\cos 30°$ and $\left|\overrightarrow{F_2}\right| = 40\cos 35°$ (This resolves the 50N and the 40 N force into their horizontal components.)

$\therefore \left|\vec{P}\right| = \left|\overrightarrow{F_1}\right|\cos 30° + \left|\overrightarrow{F_2}\right|\cos 30°$

$= (50\cos 30°)\cos 30° + (40\cos 35°)\cos 30°$ (This combines the two forces into one force along the direction of travel.)

$= 50\cos^2 30° + 40\cos 35°\cos 30°$

$= 50(0.75) + 40(0.7094)$

$= 37.5 + 28.4$

$\doteq 65.9$

The force pulling the snowmobile forward is approximately 65.9 N.

- Find the force that is tending to lift the snowmobile upward.

Solution:

Let the force tending to lift the snowmobile be $\vec{Q}$. (Refer to the previous diagram.)

$$\left|\vec{Q}\right| = 50 \sin 30° + 40 \sin 35°$$
$$= 50(0.5) + 40(0.5736)$$
$$= 25.0 + 22.9$$
$$\doteq 47.9$$

(Resolve each force into its vertical component. Because each is a lifting force, simply add the forces.)

The force tending to lift the snowmobile is 47.9 N.

You should try another example. This time you will find the angle of incline.

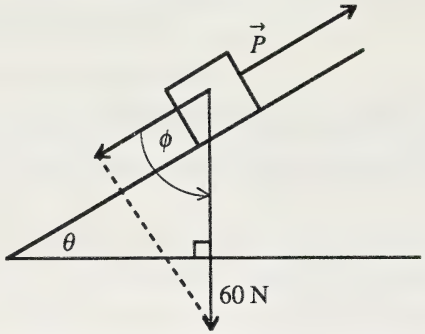
Example 3

A force of 46 N was applied in the direction of a ramp to support a weight of 60 N on the smooth surface of the ramp.

- What is the angle of incline of the ramp if the system is in equilibrium? (Round to the nearest degree.)

Solution:

First you draw and label a diagram.



Since the weight of the object is 60 N vertically downward, then the force $\vec{P}$ will keep the system in equilibrium if the incline slope is at an angle θ. To find the angle of incline, first you must find the angle ϕ between the 60 N vertical force and the component parallel to the plane.

$$\left|\vec{P}\right| = 60 \cos \phi$$
$$\therefore 46 = 60 \cos \phi$$
$$\cos \phi = \frac{46}{60}$$
$$= 0.7667$$
$$\phi \doteq 40°$$
$$\therefore \theta = 90° - \phi \doteq 50°$$

The angle of incline θ is 50°.

- What support force would you need if angle θ of the inclined plane were 45°? (Round to one decimal place.)

Solution:

If θ were 45°, then the angle between the vertical and the line parallel to the incline would also be 45° $(90° - 45° = 45°)$.

$$\therefore \left|\vec{P}\right| = 60 \cos 45°$$

$$= 60\left(\frac{1}{\sqrt{2}}\right)$$

$$= 60\left(\frac{\sqrt{2}}{2}\right)$$

$$= 30\sqrt{2}$$

$$= 30(1.4142)$$

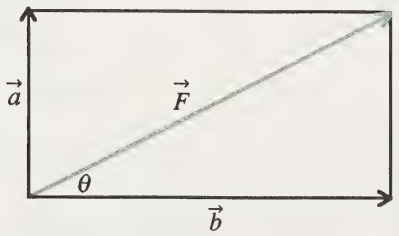
$$= 42.426$$

$$\doteq 42.4$$

If the incline of the ramp were 45°, you would need a force of approximately 42.4 N to keep the weight from sliding down.

Now try the following questions. (Do any five.)

1. In the following diagram $\vec{a}$ and $\vec{b}$ are rectangular components of the force $\vec{F}$. Find $\left|\vec{a}\right|$ and $\left|\vec{b}\right|$ for the given values of $\left|\vec{F}\right|$ and θ. (Give answers to the nearest unit or in radical form.)



   a. $\left|\vec{F}\right| = 90$ N, $\theta = 60°$

   b. $\left|\vec{F}\right| = 160$ N, $\theta = 45°$

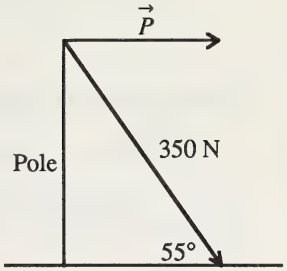
2. A trunk weighing 400 N rests on a smooth inclined plane at an angle of 35° to the horizontal. Find the force being applied parallel to the plane to keep the trunk stationary on the ramp.

3. Two people are pulling a sleigh along a smooth path of snow. Each person is exerting a pull of 30 N at a certain angle θ to the horizontal and at an angle of 30° on either side of the path of the sleigh. If the force drawing the sleigh forward is 45 N, then answer the following questions.

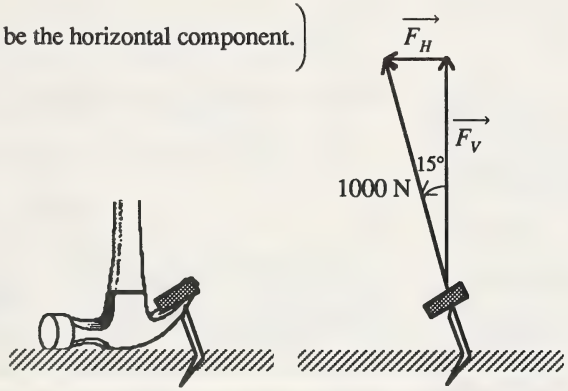
   a. What is the angle θ?

   b. What is the upward force tending to lift the sleigh?

4. A telephone pole is anchored by means of a guy wire as shown in the following diagram. What is the magnitude of the force $\vec{P}$ perpendicular to the pole if the tension in the guy wire is 350 N and the wire makes an angle of 55° with the ground?

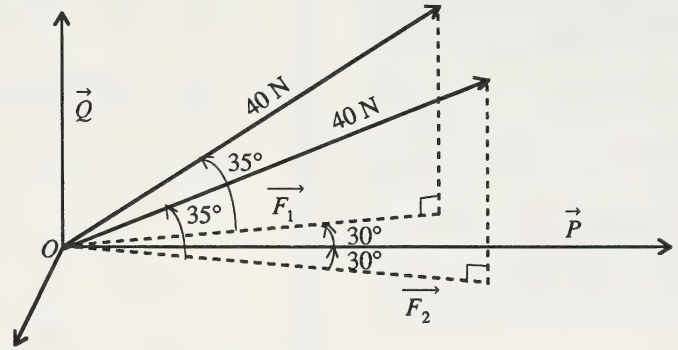


5. In removing a nail a force of 1000 N is applied by a hammer in the direction shown below. What are the horizontal and vertical components? (Note: Let $\overrightarrow{F_V}$ be the vertical component and $\overrightarrow{F_H}$ be the horizontal component.)



6. Two people are pulling a toboggan along smooth, level ice. Each person is exerting a pull of 40 N at an angle of 35° to the horizontal and at an angle of 30° on either side of the direction of travel. Find the force that draws the toboggan forward and the force tending to lift the toboggan.

Let the force drawing the toboggan forward be $\vec{P}$, the force tending to lift the toboggan be $\vec{Q}$, and the horizontal components of the applied forces be $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$.



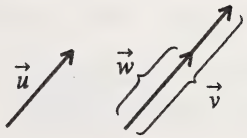
For solutions to **Activity 1**, turn to the **Appendix, Topic 3**.

Activity 2

Define and apply linear combinations of geometric vectors.

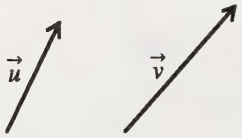
Collinear Vectors

If you consider two nonzero parallel vectors $\vec{u}$ and $\vec{v}$, it is possible to cut off on $\vec{v}$ a vector $\vec{w}$ equal to $\vec{u}$ as illustrated in the following diagram.



Now $\vec{u}$ and $\vec{w}$ are equal vectors since they are equal in magnitude and direction. Also, for some scalar k, $\vec{v} = k\vec{w} = k\vec{u}$. You say $\vec{v}$ is a scalar multiple of $\vec{w}$, and $\vec{u}$ is a scalar multiple of $\vec{v}$.

On the other hand if two vectors $\vec{u}$ and $\vec{v}$ are nonparallel as illustrated in the following diagram, it is impossible to express one as the scalar multiple of the other. The directions of $k\vec{u}$ and $\vec{u}$ are the same if $k > 0$ and opposite if $k < 0$. Thus, $\vec{v} \neq k\vec{u}$ for any scalar k, although it is possible to find a vector $k\vec{u}$ such that $\left|\vec{v}\right| = \left|k\vec{u}\right|$.



Two vectors are said to be **coplanar** if they are both contained in the same plane.

Two coplanar vectors are said to be **collinear** if one vector is a scalar multiple of the other.

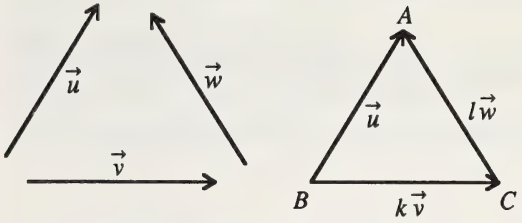
This means that collinear vectors may or may not be situated in a straight line, but they can be translated into a straight line.

If $\vec{v} = k\vec{u}$, then $\frac{1}{k}\vec{v} = \vec{u}$ (dividing both sides of an equation by k).

Therefore, $\vec{u}$ is a scalar multiple of $\vec{v}$.

Linear Combinations

Consider three coplanar vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ that are also noncollinear. You can use these vectors to form a triangle as shown in the following diagram.



Now ΔABC represents a force triangle of the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ such that $\overrightarrow{BA} = \vec{u}$, $\overrightarrow{BC} = k\vec{v}$, and $\overrightarrow{CA} = l\vec{w}$.

This means that $\overrightarrow{BA}$ is parallel to $\vec{u}$, $\overrightarrow{BC}$ is parallel to $\vec{v}$, and $\overrightarrow{CA}$ is parallel to $\vec{w}$.

From your knowledge of vector addition, you know that $\overrightarrow{BA} = \overrightarrow{BC} + \overrightarrow{CA}$ or that $\vec{u} = k\vec{v} + l\vec{w}$ (where k and l are scalars and $k, l \neq 0$).

If this can be shown for any three coplanar, noncollinear vectors, then you have a **linear combination**.

You can express $\vec{u}$ as a linear combination of $\vec{v}$ and $\vec{w}$.

Since k and l are nonzero quantities, $\vec{v}$ and $\vec{w}$ may be expressed as follows:

$$\vec{v} = \frac{1}{k}\vec{u} - \frac{l}{k}\vec{w} \quad \left(\vec{v} \text{ as a linear combination of } \vec{u} \text{ and } \vec{w}\right)$$

$$\vec{w} = \frac{1}{l}\vec{u} - \frac{k}{l}\vec{v} \quad \left(\vec{w} \text{ as a linear combination of } \vec{u} \text{ and } \vec{v}\right)$$

In general when you multiply two vectors each by a scalar such that the sum of the two vectors equals a third vector, then the third vector is a linear combination of the first two.

A vector can also be a linear combination of more than two vectors. For example, when you have three coplanar vectors, two of which are collinear, you have a special situation.

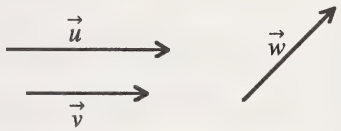
If $\vec{u} = k\vec{v} + l\vec{w}$, then you can solve this equation for $\vec{v}$ as follows:

$$\vec{u} - l\vec{w} = k\vec{v} \quad \left(\begin{array}{l}\text{subtracting } l\vec{w} \\ \text{from each side}\end{array}\right)$$

$$\therefore \frac{1}{k}\vec{u} - \frac{l}{k}\vec{w} = \vec{v} \quad \text{(dividing by } k\text{)}$$

Similarly, you can solve the original equation for $\vec{w}$.

In the following diagram $\vec{w}$ is not parallel to $\vec{u}$ and $\vec{v}$, where $\vec{u}$ and $\vec{v}$ are parallel (collinear) vectors. Now $\vec{u}$ is a linear combination of $\vec{v}$ and $\vec{w}$, and $\vec{v}$ is a linear combination of $\vec{u}$ and $\vec{w}$. However, $\vec{w}$ is **not** a linear combination of $\vec{u}$ and $\vec{v}$.



For a scalar k where $k>1$, then $\vec{u}=k\vec{v}$ since $\vec{u}$ and $\vec{v}$ are collinear.

$$\therefore \vec{u} = k\vec{v} + \vec{0}$$
$$= k\vec{v} + 0\vec{w}$$
$$\therefore \vec{v} = \frac{1}{k}\vec{u} + 0\vec{w}$$

In general a vector cannot be a linear combination of two collinear vectors unless it is also collinear with them.
However, a vector can be a linear combination of two vectors and be collinear with only one of them.

Look at some examples to see how you can express a vector as a linear combination of two other vectors.

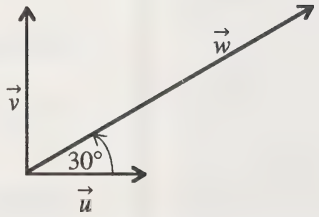
Example 4

$\vec{u}$, $\vec{v}$, and $\vec{w}$ are three coplanar vectors making angles 0°, 90°, and 30° respectively with the horizontal. If $|\vec{u}|=4$, $|\vec{v}|=6$, and $|\vec{w}|=12$, express $\vec{w}$ as a linear combination of $\vec{u}$ and $\vec{v}$.

Solution:

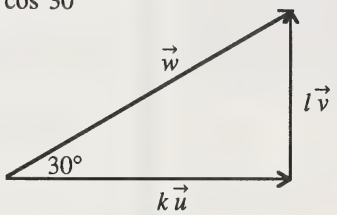
First draw a diagram.

Let $\vec{w}=k\vec{u}+l\vec{v}$.



Then, from this diagram the following may be stated:

$$|k\vec{u}| = |\vec{w}| \cos 30°$$



$$\therefore 4k = 12\left(\frac{\sqrt{3}}{2}\right) \quad \text{(by substitution)}$$

$$k = \frac{(12)(\sqrt{3})}{(4)(2)}$$

$$= \frac{3\sqrt{3}}{2}$$

From the diagram, $\left|l\vec{v}\right| = \left|\vec{w}\right| \sin 30°$.

$$\therefore 6l = 12\left(\frac{1}{2}\right)$$

$$l = \frac{6}{6}$$

$$= 1$$

$$\therefore \vec{w} = \frac{3\sqrt{3}}{2}\vec{u} + \vec{v}$$

By using properties realted to vectors, $\vec{v}$ and $\vec{u}$ can be expressed as a linear combination of the other vectors.

$$\vec{v} = \vec{w} - \frac{3\sqrt{3}}{2}\vec{u}$$

$$\vec{u} = \frac{2}{3\sqrt{3}}\vec{w} - \frac{2}{3\sqrt{3}}\vec{v}$$

$$= \frac{2\sqrt{3}}{9}\vec{w} - \frac{2\sqrt{3}}{9}\vec{v}$$

Verification:

If $\vec{w} = \frac{3\sqrt{3}}{2}\vec{u} + \vec{v}$, then you can state the following by the Pythagorean theorem:

$$\left|\vec{w}\right|^2 = \left[\left(\frac{3\sqrt{3}}{2}\right)(4)\right]^2 + (6)^2$$

$$= \left(6\sqrt{3}\right)^2 + (6)^2$$

$$= 36(3) + 36$$

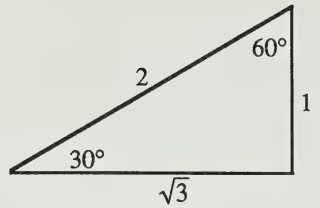
$$= 144$$

Given $\left|\vec{w}\right| = 12$, then $\left|\vec{w}\right|^2 = 144$ is correct.

Line Division Method for Linear Combinations

Now you will look at a method to find the linear combination of two vectors. Three collinear points P, Q, and R with Q between P and R, and O a point not on the line determined by P, Q, and R are shown in the following diagram.

The cosine of 30° is exact. It was determined from the basic 30°-60°-90° triangle.



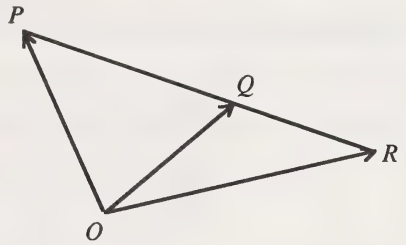
Here $\cos 30° = \frac{\sqrt{3}}{2}$ and $\sin 30° = \frac{1}{2}$.

Here the denominators are rationalized.

$$\frac{2}{3\sqrt{3}} = \left(\frac{2}{3\sqrt{3}}\right)\left(\frac{\sqrt{3}}{\sqrt{3}}\right)$$

$$= \frac{2\sqrt{3}}{(3)(3)}$$

$$= \frac{2\sqrt{3}}{9}$$



Since a line and a point not on the line determine a plane, the vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$ are coplanar; therefore, any one of these vectors may be expressed as a linear combination of the other two.

The following examples will illustrate how you can express one vector as a linear combination of the other two vectors where all three are coplanar.

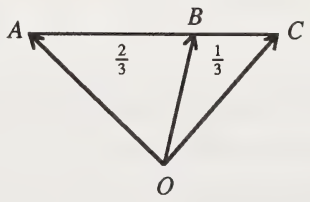
Example 5

A, B, and C are collinear points with B between A and C such that $AB:BC = 2:1$. O is a point not on the line containing A, B, and C. Prove that $\overrightarrow{OB} = \frac{2}{3}\overrightarrow{OC} + \frac{1}{3}\overrightarrow{OA}$.

Solution:

First draw a diagram.

The ratios $\frac{2}{3}$ and $\frac{1}{3}$ are obtained from the ratio of each of the lengths of segments AB and BC to the total length of segment AC.



Now you will see that $\overrightarrow{OB}$ can be expressed as a linear combination of $\overrightarrow{OA}$ and $\overrightarrow{OC}$.

You know that $AB:BC = 2:1$; therefore, $AB = 2BC$. (1)

Now express segments AB and BC in terms of $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$.

Given $AB = 2$ and $BC = 1$, then $AC = 3$.

Therefore, the ratios $\frac{AB}{AC} = \frac{2}{3}$ and $\frac{BC}{AC} = \frac{1}{3}$.

If $AB:BC = 2:1$, then $\frac{AB}{BC} = \frac{2}{1}$ and $AB = 2BC$.

The following may be stated from the diagram:

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$ (vector addition)

$\therefore \overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$ (2) (inverse vectors)

$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$ (vector addition)

$\therefore \overrightarrow{BC} = -\overrightarrow{OB} + \overrightarrow{OC}$ (3) (inverse vectors)

Now substitute the information from equations (2) and (3) into equation (1).

$$-\overrightarrow{OA} + \overrightarrow{OB} = 2\left(-\overrightarrow{OB} + \overrightarrow{OC}\right)$$

$$-\overrightarrow{OA} + \overrightarrow{OB} = -2\overrightarrow{OB} + 2\overrightarrow{OC}$$

$$\overrightarrow{OB} + 2\overrightarrow{OB} = 2\overrightarrow{OC} + \overrightarrow{OA}$$

$$3\overrightarrow{OB} = 2\overrightarrow{OC} + \overrightarrow{OA}$$

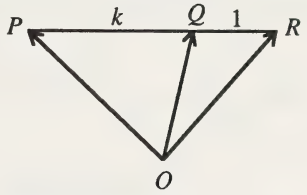
$$\overrightarrow{OB} = \frac{2}{3}\overrightarrow{OC} + \frac{1}{3}\overrightarrow{OA}$$

Therefore, $\overrightarrow{OB}$ is a linear combination of $\overrightarrow{OC}$ and $\overrightarrow{OA}$ written as $\overrightarrow{OB} = \frac{2}{3}\overrightarrow{OC} + \frac{1}{3}\overrightarrow{OA}$.

Be sure to review the algebraic manipulations. Note how to use the information that is given. Segment AB is twice as large as segment BC. Now study another example where the ratio of the segments contains a variable.

Example 6

If P, Q, and R are collinear and $PQ:QR = k:1$, then express $\overrightarrow{OQ}$ as a linear combination of $\overrightarrow{OP}$ and $\overrightarrow{OR}$ where O is a point not on the line containing P, Q, and R, and Q is between P and R.



$$PR = PQ + QR$$
$$= k + 1$$
$$\therefore \frac{PQ}{PR} = \frac{k}{k+1}$$
$$\frac{QR}{PR} = \frac{1}{k+1}$$

Since $\overrightarrow{OB} = \frac{2}{3}\overrightarrow{OC} + \frac{1}{3}\overrightarrow{OA}$, you can express $\overrightarrow{OC}$ as a linear combination of $\overrightarrow{OB}$ and $\overrightarrow{OA}$ by manipulating this equation. This could also be done to express $\overrightarrow{OA}$ as a linear combination of $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

Solve for $\overrightarrow{OA}$:

$$\overrightarrow{OB} = \frac{2}{3}\overrightarrow{OC} + \frac{1}{3}\overrightarrow{OA}$$
$$3\overrightarrow{OB} = 2\overrightarrow{OC} + \overrightarrow{OA}$$
$$\therefore \overrightarrow{OA} = 3\overrightarrow{OB} - 2\overrightarrow{OC}$$

Solve for $\overrightarrow{OC}$:

$$\overrightarrow{OC} = \frac{3\overrightarrow{OB} - \overrightarrow{OA}}{2}$$
$$= \frac{3}{2}\overrightarrow{OB} - \frac{1}{2}\overrightarrow{OA}$$

Solution:

P, Q, and R are collinear and $PQ = k\,QR$.

$$\therefore \overrightarrow{PO} + \overrightarrow{OQ} = k\left(\overrightarrow{QO} + \overrightarrow{OR}\right)$$

$$-\overrightarrow{OP} + \overrightarrow{OQ} = k\left(-\overrightarrow{OQ} + \overrightarrow{OR}\right)$$

$$-\overrightarrow{OP} + \overrightarrow{OQ} = -k\overrightarrow{OQ} + k\overrightarrow{OR}$$

$$k\overrightarrow{OQ} + \overrightarrow{OQ} = k\overrightarrow{OR} + \overrightarrow{OP}$$

$$(k+1)\overrightarrow{OQ} = k\overrightarrow{OR} + \overrightarrow{OP}$$

$$\therefore \overrightarrow{OQ} = \frac{k}{k+1}\overrightarrow{OR} + \frac{1}{k+1}\overrightarrow{OP}$$

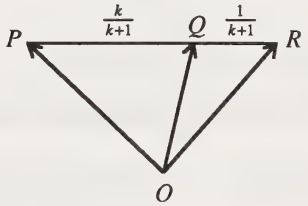
There is another way of doing this example which you may find easier.

The following three steps can be used to express $\overrightarrow{OQ}$ as a linear combination of $\overrightarrow{OP}$ and $\overrightarrow{OR}$.

Step 1: You must determine the fractions that the parts of the divided side are to the whole side.

The given ratio $PQ:QR = k:1$ means $PQ = k$ and $QR = 1$. Since $PR = PQ + QR$, then $PR = k + 1$; thus, $\frac{PQ}{PR} = \frac{k}{k+1}$ and $\frac{QR}{PR} = \frac{1}{k+1}$.

Here the parts are $\frac{k}{k+1}$ and $\frac{1}{k+1}$ as shown in the diagram.



Step 2: Multiply $\overrightarrow{OR}$ by $\frac{k}{k+1}$ and $\overrightarrow{OP}$ by $\frac{1}{k+1}$.

Step 3: Thus, $\overrightarrow{OQ} = \frac{k}{k+1}\overrightarrow{OR} + \frac{1}{k+1}\overrightarrow{OP}$.

To express $\overrightarrow{OP}$ or $\overrightarrow{OR}$ as a linear combination of the other two vectors, algebraically manipulate this equation.

Note the following from the diagram:

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$$

$$= -\overrightarrow{OP} + \overrightarrow{OQ}$$

$$\overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR}$$

$$= -\overrightarrow{OQ} + \overrightarrow{OR}$$

Note that $\overrightarrow{OQ}$ is sometimes called the **divider**.

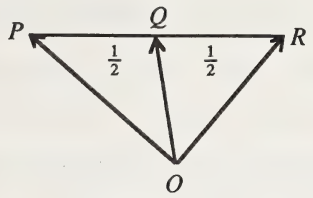
This simpler method is used in the following examples.

Example 7

The segment PR is cut in half by the point Q within it. Express $\overrightarrow{OQ}$ as a linear combination of $\overrightarrow{OP}$ and $\overrightarrow{OR}$.

Solution:

First draw a diagram expressing the given information. Point Q is between P and R.

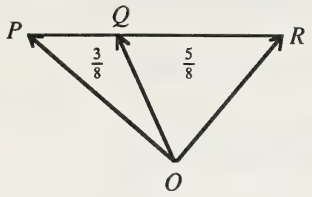


Therefore, $\overrightarrow{OQ} = \frac{1}{2}\overrightarrow{OP} + \frac{1}{2}\overrightarrow{OR}$.

Example 8

Point Q cuts segment PR in the ratio $\frac{3}{5}$ – that is, $PQ:QR = 3:5$. Express $\overrightarrow{OR}$ as a linear combination of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.

Solution:



Here $\overrightarrow{OR}$ is not the divider.

Express the divider, namely $\overrightarrow{OQ}$, as a linear combination of $\overrightarrow{OP}$ and $\overrightarrow{OR}$. Then, algebraically manipulate the equation to express $\overrightarrow{OR}$ as a linear combination of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.

$$\overrightarrow{OQ} = \frac{3}{8}\overrightarrow{OR} + \frac{5}{8}\overrightarrow{OP}$$

$$8\overrightarrow{OQ} = 3\overrightarrow{OR} + 5\overrightarrow{OP}$$

$$3\overrightarrow{OR} = -5\overrightarrow{OP} + 8\overrightarrow{OQ}$$

$$\overrightarrow{OR} = -\frac{5}{3}\overrightarrow{OP} + \frac{8}{3}\overrightarrow{OQ}$$

The divider is the vector inside ΔPOR that divides side PR.

Example 9

If $\overrightarrow{OA} = \vec{u}$, $\overrightarrow{OB} = \vec{v}$, and $\overrightarrow{OC} = \vec{w}$ such that A, B, and C are collinear and $AB:BC = 5:-3$, then express $\vec{v}$ as a linear combination of $\vec{u}$ and $\vec{w}$.

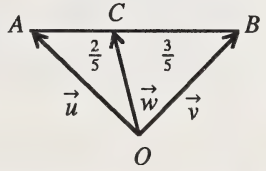
Solution:

Draw a diagram and label it. Notice in this example that one of the parts of the ratio is negative. This means you must be careful of the direction when drawing the line and labelling the points. Since segment AB is given as positive, you draw and label it from left to right. On the other hand segment BC is given as negative directed so you draw and label it from right to left. (See the margin box.) Using the simple method you can express the divider $\vec{w}$ as a linear combination of $\vec{u}$ and $\vec{v}$.

$$\vec{w} = \frac{2}{5}\vec{v} + \frac{3}{5}\vec{u}$$

$$5\vec{w} - 3\vec{u} = 2\vec{v}$$

$$\frac{5}{2}\vec{w} - \frac{3}{2}\vec{u} = \vec{v}$$

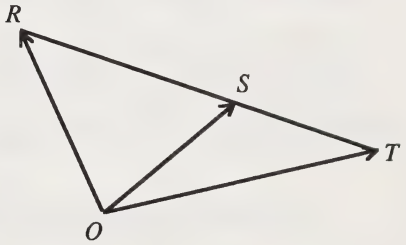


Thus, $\vec{v}$ expressed as a linear combination of $\vec{u}$ and $\vec{w}$ is $\vec{v} = \frac{5}{2}\vec{w} - \frac{3}{2}\vec{u}$.

You can also show that three points are collinear if you are given the linear combination of the vectors from O to the three points.

Example 10

In the following diagram R, S, and T are collinear and O is a point not on segment RT. If $\overrightarrow{OT} = \frac{4}{3}\overrightarrow{OS} - \frac{1}{3}\overrightarrow{OR}$, then use vector properties to determine the ratio $RS:ST$ and show that R, S, and T are collinear.



Recall that two vectors are collinear if they are scalar multiples of each other.

Notice that $AB:BC = 5:-3$. This means that segment AB is positive so you draw AB from left to right. Segment BC is negative so it is from right to left. Thus, C is between A and B.

$$\left|\overrightarrow{AB}\right| = 5$$

$$\left|\overrightarrow{BC}\right| = 3$$

$$\therefore \left|\overrightarrow{AC}\right| = 2$$

Therefore, the ratios for the diagram are $\frac{2}{5}$ and $\frac{3}{5}$.

Solution:

$$\overrightarrow{ST} = \overrightarrow{SO} + \overrightarrow{OT}$$

$$= -\overrightarrow{OS} + \left(\frac{4}{3}\overrightarrow{OS} - \frac{1}{3}\overrightarrow{OR}\right)$$

$$= -\frac{1}{3}\overrightarrow{SO} - \frac{1}{3}\overrightarrow{OR}$$

$$= -\frac{1}{3}\left(\overrightarrow{SO} + \overrightarrow{OR}\right) \quad (1)$$

$$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR}$$

$$= -\overrightarrow{SO} - \overrightarrow{OR}$$

$$= -\left(\overrightarrow{SO} + \overrightarrow{OR}\right) \quad (2)$$

Combining (1) and (2) you may state the following:

$$\frac{\overrightarrow{RS}}{\overrightarrow{ST}} = \frac{-\left(\overrightarrow{SO} + \overrightarrow{OR}\right)}{-\frac{1}{3}\left(\overrightarrow{SO} + \overrightarrow{OR}\right)} = \frac{3}{1}$$

$$\therefore \frac{RS}{ST} = \frac{3}{1}$$

Since $\overrightarrow{RS} = 3\overrightarrow{ST}$, then R, S, and T are collinear and $RS:ST = 3:1$.

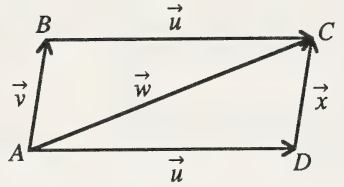
You can also use vector methods to prove statements concerning geometric figures.

Example 11

If two opposite sides of quadrilateral $ABCD$ are equal and parallel, then prove that the other two opposite sides are also equal and parallel.

Solution:

Given $\overrightarrow{BC} \parallel \overrightarrow{AD}$ and $\overrightarrow{BC} = \overrightarrow{AD}$, let $\overrightarrow{BC} = \overrightarrow{AD} = \vec{u}$. Now prove $\overrightarrow{AB} \parallel \overrightarrow{DC}$ and $\overrightarrow{AB} = \overrightarrow{DC}$. (In other words prove that $\vec{v} = \vec{x}$.)



First join A and C, and let $\overrightarrow{AC} = \vec{w}$.

Then $\vec{v} + \vec{u} = \vec{w}$ from ΔABC, and $\vec{u} + \vec{x} = \vec{w}$ from ΔADC.

Since $\vec{w} = \vec{w}$, the following may be stated:

$\vec{v} + \vec{u} = \vec{u} + \vec{x}$

$\therefore \vec{v} = \vec{x}$

Since $\vec{v} = \vec{x}$, they are equivalent; hence,

$\overrightarrow{AB} \parallel \overrightarrow{DC}$ and $\overrightarrow{AB} = \overrightarrow{DC}$.

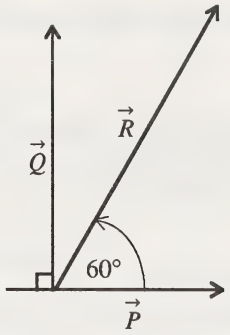
For a summary of the concepts of geometric vectors you may wish to view the video titled **Geometric Vectors**. This video is program 13 in the *Catch 31*[1] series.

Now try any four of the following questions.

1. In the following diagram $\vec{P}$, $\vec{Q}$, and $\vec{R}$ represent 3 coplanar vectors where $\left|\vec{P}\right| = 10$, $\left|\vec{Q}\right| = 12$, and $\left|\vec{R}\right| = 15$.

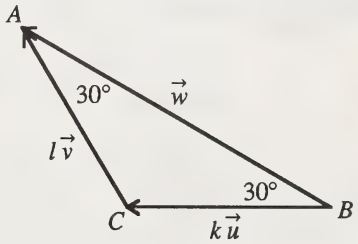
   a. Express $\vec{R}$ as a linear combination of $\vec{P}$ and $\vec{Q}$.

   b. Express $\vec{Q}$ as a linear combination of $\vec{P}$ and $\vec{R}$.



2. Examine the following triangle where $\overrightarrow{BA} = \vec{w}$, $\overrightarrow{BC} = k\vec{u}$, and $\overrightarrow{CA} = l\vec{v}$.

   If $\left|\vec{u}\right| = 4$, $\left|\vec{v}\right| = 8$, and $\left|\vec{w}\right| = 20$, use the sine law to express $\vec{w}$ as a linear combination of $\vec{u}$ and $\vec{v}$. (Note that l, $k \neq 0$.)

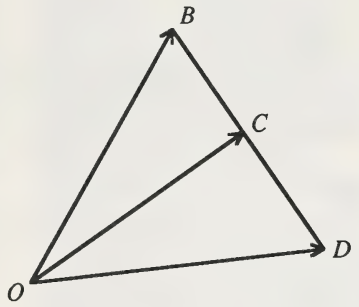


[1] *Catch 31* is a title of ACCESS Network.

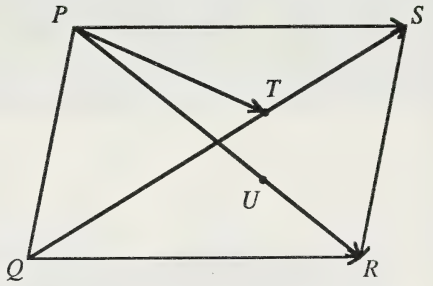
Equivalent vectors have the same magnitude and direction.

3. Q cuts segment PR into portions which are $\frac{1}{3}$ and $\frac{2}{3}$ of segment PR respectively. If $\overrightarrow{OP} = \vec{u}$, $\overrightarrow{OR} = \vec{v}$, and $\overrightarrow{OQ} = \vec{w}$, then express $\vec{v}$ as a linear combination of $\vec{u}$ and $\vec{w}$ if P, Q, and R are collinear and O is a point not on segment PR.

4. Show that B, C, and D are collinear and $BC:CD = 4:3$ if $\overrightarrow{OB} = \frac{7}{3}\overrightarrow{OC} - \frac{4}{3}\overrightarrow{OD}$.



5. Complete the steps showing that the diagonals of the parallelogram $PQRS$ bisect each other, given that T is the midpoint of $\overrightarrow{QS}$ and U is the midpoint of $\overrightarrow{PR}$.



Complete the following:

Steps:	Reason:
$\overrightarrow{PT} = \frac{1}{2}\left(\overrightarrow{PQ} + \overrightarrow{PS}\right)$	(line division method)
$= \frac{1}{2}\left(\overrightarrow{PQ} + \overrightarrow{QR}\right)$	_____
$=$ _____	$\left(\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}\right)$
$=$ _____	(U is the midpoint of $\overrightarrow{PR}$.)

Conclusion:

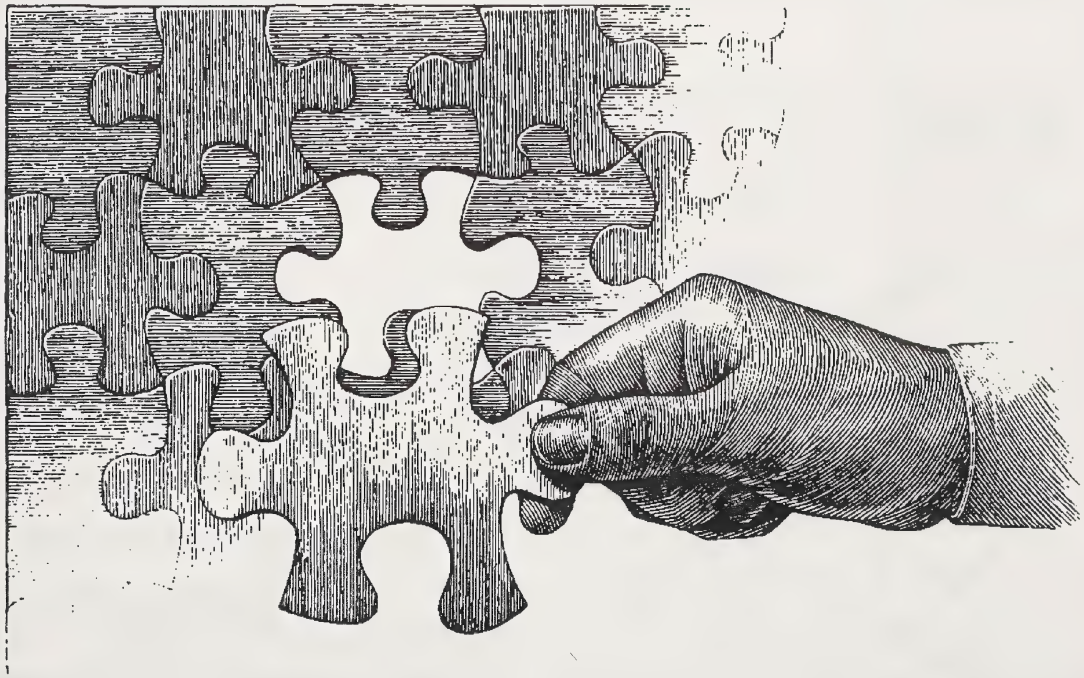
_____

_____

6. The segment AB is cut into portions which are $\frac{3}{4}$ and $\frac{1}{4}$ respectively by point P located between A and B.

   Express $\overrightarrow{OP}$ as a linear combination of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ if O is a point not collinear to segment AB.

For solutions to **Activity 2**, turn to the **Appendix, Topic 3**.



If you require help, do the Extra Help section.

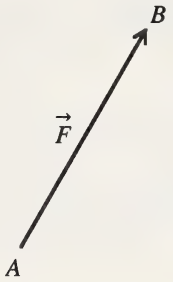
If you want more challenging explorations, do the Extensions section. } You may decide to do both.

Extra Help

Forces

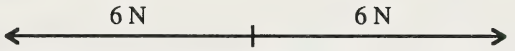
In order to represent a force, the magnitude of the force, the direction of the force, and the point of application of the force must be ascertained. These three things may be represented by a line of force called a vector.

For a visual picture the ray AB will suffice for a vector.



The vector $\overrightarrow{AB}$ represents a force $\vec{F}$ acting at point A in the direction of $\overrightarrow{AB}$ and with a magnitude represented by the length of $\overrightarrow{AB}$.

The resultant of two forces acting at a point is the single force which has the same effect at the point as the original two forces together. The following illustrations demonstrate this.



(The resultant is 0 N.)

(The resultant is 2 N to the right.)

(The resultant is 8 N.)

The equilibrant of two forces acting at a point is the single force that would produce equilibrium when added to the system of forces at that point. It is equal in magnitude and opposite in direction to the resultant.

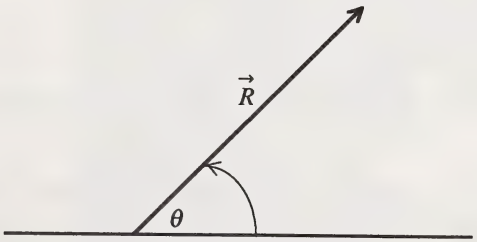
Vector forces can be added and subtracted. When you add two vectors, the answer you get is the resultant – that is, if $\vec{A}$, $\vec{B}$, and $\vec{R}$ are vectors, then $\vec{A} + \vec{B} = \vec{R}$.

However, if you are given $\vec{R}$, then how can you find $\vec{A}$ and $\vec{B}$?

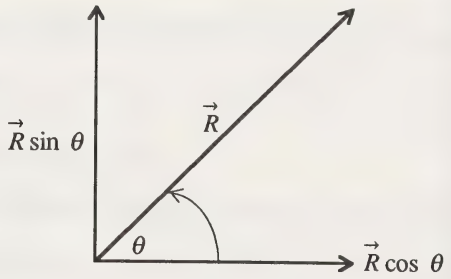
In order to do this you have to resolve the force $\vec{R}$. In other words you have to break up the resultant force into its component parts.

Consider a force $\vec{R}$. It has a magnitude and a direction.

The magnitude is $\left|\vec{R}\right|$, and the direction is θ to the horizontal.



In order to resolve this force into its vertical and horizontal components, you must split $\vec{R}$ as shown in the following diagram.



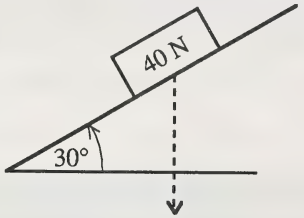
$\vec{R}\sin\theta$ is the vertical component, and $\vec{R}\cos\theta$ is the horizontal component. You will notice that $\left|\vec{R}\right| = \left|\vec{R}\sin\theta\right| + \left|\vec{R}\cos\theta\right|$.

This is a simple case. You can resolve a vector into as many components as you want.

Study the next example where you resolve a force into its components.

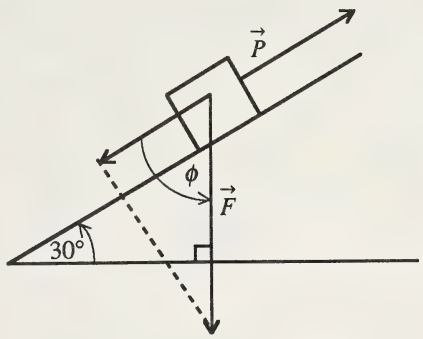
Example 12

A 40 N block of ice rests on a smooth ramp which is inclined at an angle of 30° to the horizontal. What upward force along the direction of the ramp is required in order to keep the ice from sliding?



Solution:

Let the required force to keep the ice at rest be $\vec{P}$.
First draw a diagram.



The force you are interested in is parallel to the incline. The 40 N force acts in a downward direction.

To find the force parallel to the plane, you need to use the following relationship:

$$\left|\vec{P}\right| = \left|\vec{F}\right| \cos\phi \qquad (\phi \text{ is the complement of the angle of the incline, and } \vec{F} \text{ is the vertical downward force exerted by the block of ice.})$$

$$= 40 \cos 60°$$

$$= 40\left(\frac{1}{2}\right)$$

$$= 20$$

Therefore, the force needed to keep the ice from sliding is 20 N.

Linear Combinations

It is interesting to note that given any three coplanar noncollinear vectors, you can express any one in terms of the other two.

When you do this, refer to the construction as a linear combination.

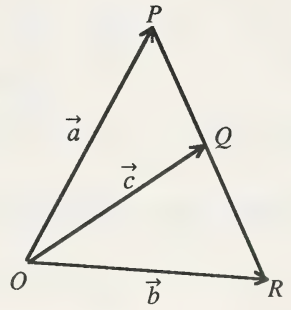
Given three coplanar vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, then you can write $\vec{c} = k\vec{a} + l\vec{b}$ where k and l are scalar quantities and $k, l \neq 0$. Here $\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$.

You can also write the following:

$$\vec{a} = \frac{\vec{c}}{k} - \frac{l\vec{b}}{k} \qquad \left(\vec{a} \text{ as a linear combination of } \vec{c} \text{ and } \vec{b}\right)$$

$$\vec{b} = \frac{\vec{c}}{l} - \frac{k\vec{a}}{l} \qquad \left(\vec{b} \text{ as a linear combination of } \vec{c} \text{ and } \vec{a}\right)$$

With reference to this diagram, you can go one step further.



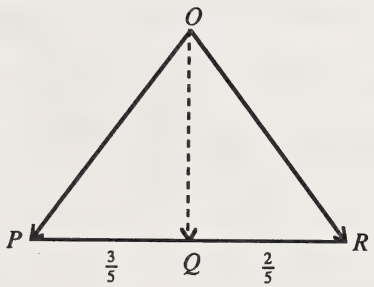
If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are three coplanar vectors with P, Q, and R being collinear and $\overrightarrow{PQ} = k\overrightarrow{QR}$, then $\vec{c} = \frac{k}{k+1}\vec{b} + \frac{1}{k+1}\vec{a}$ or $\vec{c} = \frac{k\vec{b}+\vec{a}}{k+1}$. This is the general condition for a linear combination.

You should note that $PQ = k\,QR$ may be written in the form $\frac{PQ}{QR} = \frac{k}{1}$ or written as $PQ{:}QR = k{:}1$. This shows that Q divides the segment PR in the ratio k to 1.

Study the following example.

Example 13

P, Q, and R are collinear points with Q located between P and R such that $PQ{:}QR = 3{:}2$. Point O is not on the line PR. Express $\overrightarrow{OQ}$ as a linear combination of $\overrightarrow{OP}$ and $\overrightarrow{OR}$.



Solution:

$$\overrightarrow{PQ} = \frac{3}{2}\overrightarrow{QR} \qquad (1)$$

Express $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ in terms of the vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$.

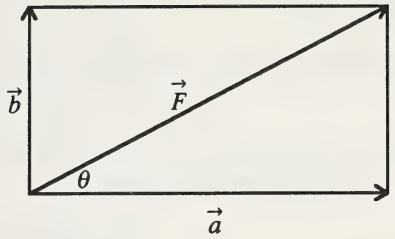
$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$$
$$= -\overrightarrow{OP} + \overrightarrow{OQ} \qquad (2)$$
$$\overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR}$$
$$= -\overrightarrow{OQ} + \overrightarrow{OR} \qquad (3)$$

Substitute information from (2) and (3) in equation (1).

$$-\overrightarrow{OP} + \overrightarrow{OQ} = \frac{3}{2}\left(-\overrightarrow{OQ} + \overrightarrow{OR}\right)$$
$$-\overrightarrow{OP} + \overrightarrow{OQ} = -\frac{3}{2}\overrightarrow{OQ} + \frac{3}{2}\overrightarrow{OR}$$
$$\frac{5}{2}\overrightarrow{OQ} = \overrightarrow{OP} + \frac{3}{2}\overrightarrow{OR}$$
$$\overrightarrow{OQ} = \frac{2}{5}\overrightarrow{OP} + \frac{3}{5}\overrightarrow{OR}$$

Now try some of the following questions. (Do any four.)

1. In the following diagram $\vec{a}$ and $\vec{b}$ are the rectangular components of the force $\vec{F}$. Find $\left|\vec{a}\right|$ and $\left|\vec{b}\right|$ for each of the following values of $\left|\vec{F}\right|$ and θ.



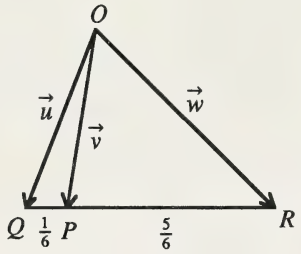
   a. $\left|\vec{F}\right| = 50$ N, $\theta = 60°$

   b. $\left|\vec{F}\right| = 96$ N, $\theta = 30°$

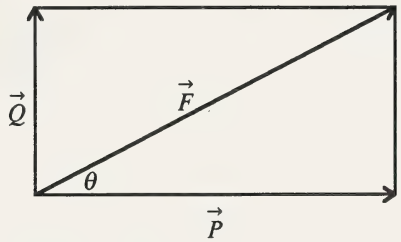
   c. $\left|\vec{F}\right| = 150$ N, $\theta = 45°$

2. Forces $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ are acting on the same point, where $\left|\overrightarrow{F_1}\right| = 60$ N and acts in an easterly direction and $\left|\overrightarrow{F_2}\right| = 80$ N and acts in a northerly direction. Calculate the magnitude and the direction of the resultant.

3. If $\overrightarrow{OP} = \vec{u}$, $\overrightarrow{OQ} = \vec{v}$, and $\overrightarrow{OR} = \vec{w}$; P, Q, and R are collinear; and $PQ{:}QR = -1{:}6$; then express $\vec{v}$ as a linear combination of $\vec{u}$ and $\vec{w}$.



4. In the following diagram $\vec{P}$ and $\vec{Q}$ are rectangular components of the force $\vec{F}$. Find $\vec{P}$ and $\vec{Q}$ for each of the following values of $\left|\vec{F}\right|$ and θ. (Give all solutions to the nearest unit.)

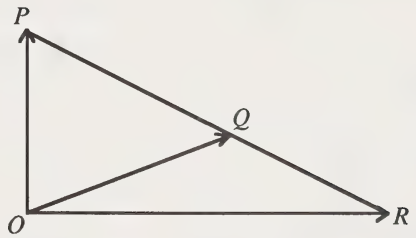


   a. $\left|\vec{F}\right| = 75$ N, $\theta = 25°$

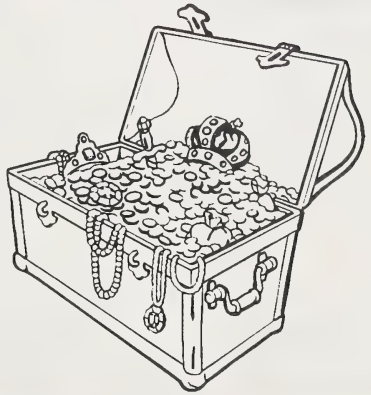
   b. $\left|\vec{F}\right| = 150$ N, $\theta = 60°$

5. A trunk weighing 500 N rests on a smooth ramp inclined at an angle of 40° to the horizontal. Find the force being applied parallel to the ramp to keep the trunk stationary on the ramp.

6. If $\overrightarrow{OP} = 3\overrightarrow{OQ} - 2\overrightarrow{OR}$, show that P, Q, and R are collinear and find the value of the ratio $PQ:QR$.



For solutions to **Extra Help**, turn to the **Appendix, Topic 3**.

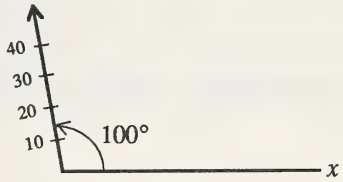


Extensions

Scientists are not the only ones who make use of vectors. In everyday life there are people who use vectors. Surveyors, ship captains, airplane pilots, and other navigators use vectors as a practice in accomplishing their tasks and in getting from one place to another.

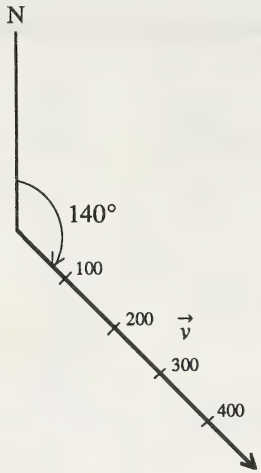
There are some minor differences in the way they go about determining the direction of the vector.

Scientists generally start from a line due east and measure the angle for the direction in a counter-clockwise manner.



This diagram could represent a force of 50 N acting at 100°. Note that the angle is measured counter-clockwise from the positive x-axis to the vector.

Most navigators and surveyors use a bearing or a heading. In their case the direction of the vector is found by measuring the angle in a clockwise manner and commencing at the due north position as shown in the following diagram.



The vector $\vec{v}$ could represent the velocity v of a plane that is travelling at 500 km/h in that direction. Note that the angle 140° is called the bearing or the heading and is measured clockwise from a line due north.

This example will give you an idea of the way navigators use vectors.

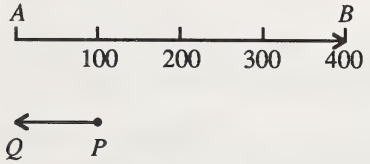
Example 14

A wind is blowing from the east at 100 km/h. Determine the resultant velocity of plane if its speed is 400 km/h and its heading is as follows. (Give the resultant to the nearest km/h and nearest degree.)

- 90°

Solution:

Let $\overrightarrow{AB}$ represent the velocity of the plane and $\overrightarrow{PQ}$ represent the velocity of the wind.



Thus, the resultant is $\overrightarrow{AB} - \overrightarrow{PQ} = 400 - 100 = 300$ km/h due east.

- 270°

Solution:

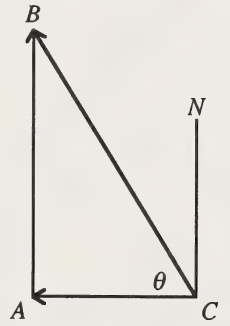
Let $\overrightarrow{AB}$ represent the velocity of the plane and $\overrightarrow{PQ}$ represent the velocity of the wind. In this case the aircraft is going in the same direction as the wind is blowing.

Thus, the resultant is $\overrightarrow{AB} + \overrightarrow{PQ} = 400 + 100 = 300$ km/h due west.

- 0°

Solution:

Use the right triangle and trigonometry to do this part. Since the heading is 0°, the plane is flying north. Let $\overrightarrow{AB}$ represent the velocity of the plane and $\overrightarrow{CA}$ represent the wind velocity.



The heading of the resultant is $270° + \theta$ where $\theta = \angle BCA$.

$$\therefore \tan \theta = \frac{\overrightarrow{AB}}{\overrightarrow{CA}}$$
$$= \frac{400}{100}$$
$$= 4$$
$$\therefore \theta = 76°$$
$$270° + 76° = 346°$$

The magnitude of $\overrightarrow{CB}$ corresponds to the resultant velocity of the plane. $\overrightarrow{CB}$ represents the magnitude of $\overrightarrow{CB}$.

From the Pythagorean theorem you can write the following:

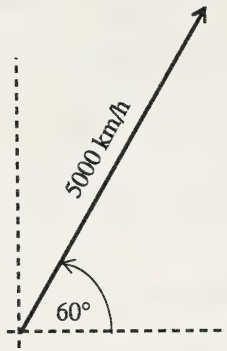
$$\left|\overrightarrow{CB}\right|^2 = \left|\overrightarrow{CA}\right|^2 + \left|\overrightarrow{AB}\right|^2$$
$$= 100^2 + 400^2$$
$$= 100^2\left(1 + 4^2\right)$$
$$= 100^2(17)$$
$$\left|\overrightarrow{CB}\right| = 100\sqrt{17}$$
$$\doteq 412$$

The resultant velocity is 412 km/h on a heading of 346°.

Look at another example.

Example 15

A rocket with a speed of 5000 km/h is ascending at an angle of 60° to the horizontal.



- What is its rate of climb?

Solution:

The rate of climb is given by the vertical component.

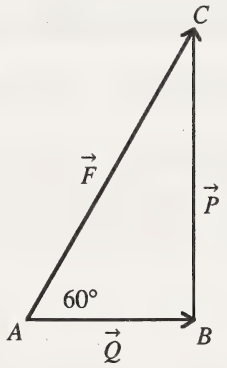
$$\therefore \left|\vec{P}\right| = \left|\vec{F}\right| \sin 60°$$

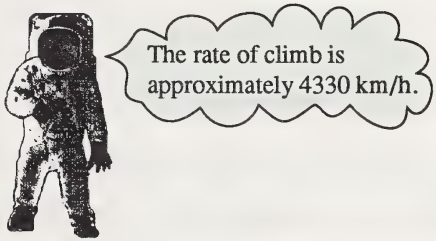
$$= 5000 \sin 60°$$

$$= 5000 \frac{\sqrt{3}}{2}$$

$$= 2500\sqrt{3}$$

$$\doteq 4330$$





- What is its horizontal ground velocity?

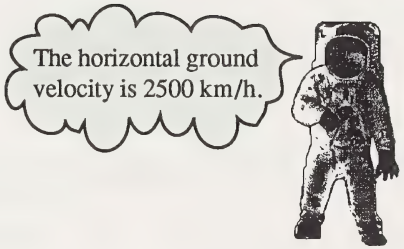
Solution:

Let the ground velocity be $\vec{Q}$.

$$\therefore \left|\vec{Q}\right| = \left|\vec{F}\right| \cos 60°$$

$$= 5000(0.5)$$

$$= 2500$$



Vectors are very useful, but there are also many intellectual exercises that are associated with them. Some of these exercises have no practical value at this time.

Work through the following.

Example 16

Simplify $2\vec{a} + \vec{b} + 3\vec{c} - \left[\vec{a} - 2\vec{b} - 2\left(2\vec{a} - 3\vec{b} - \vec{c}\right)\right]$.

Solution:

$$2\vec{a} + \vec{b} + 3\vec{c} - \left[\vec{a} - 2\vec{b} - 2\left(2\vec{a} - 3\vec{b} - \vec{c}\right)\right]$$

$$= 2\vec{a} + \vec{b} + 3\vec{c} - \vec{a} + 2\vec{b} + 2\left(2\vec{a} - 3\vec{b} - \vec{c}\right)$$

$$= 2\vec{a} + \vec{b} + 3\vec{c} - \vec{a} + 2\vec{b} + 4\vec{a} - 6\vec{b} - 2\vec{c}$$

$$= 5\vec{a} - 3\vec{b} + \vec{c}$$

Example 17

A force of 12 N is resolved into two components of 8 N and 7 N. Find the angle which each component makes with the 12 N force if the components are non-rectangular.

Solution:

Draw a diagram.

Let the angle between $\overrightarrow{PR}$ and $\overrightarrow{PQ}$ be θ.

$$\left|\overrightarrow{QR}\right|^2 = \left|\overrightarrow{PR}\right|^2 + \left|\overrightarrow{PQ}\right|^2 - 2\left|\overrightarrow{PR}\right|\left|\overrightarrow{PQ}\right|\cos\theta \quad \text{(cosine law)}$$

$$\therefore 7^2 = 12^2 + 8^2 - 2(12)(8)\cos\theta$$

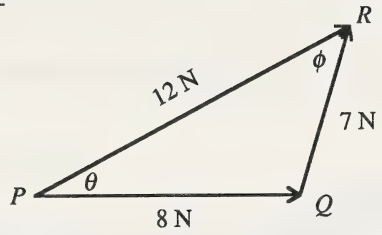
$$\cos\theta = \frac{12^2 + 8^2 - 7^2}{2(12)(8)}$$

$$= \frac{144 + 64 - 49}{192}$$

$$= \frac{159}{192}$$

$$= 0.8281$$

$$\therefore \theta \doteq 34°$$



The angle between the 12 N force and the 8 N force is 34°.

Let the angle between $\overrightarrow{PR}$ and $\overrightarrow{QR}$ be ϕ.

$$8^2 = 7^2 + 12^2 - 2(7)(12)\cos\phi \quad \text{(cosine law)}$$

$$2(7)(12)\cos\phi = 7^2 + 12^2 - 8^2$$

$$\cos\phi = \frac{49 + 144 - 64}{2(84)}$$

$$= \frac{193 - 64}{168}$$

$$= \frac{129}{168}$$

$$= 0.7679$$

$$\therefore \phi \doteq 40°$$

The angle between the 12 N force and the 7 N force is 40°.

Now try the following questions. (Do any five.)

1. In opening a vertical window from the top by means of a window pole, a man pulls with a force of 50 N. How much of the man's force actually lowers the window if the pole makes an angle of 25° with the window? Assume the window slides vertically downward.

2. How far north will an aeroplane fly in one hour if it has an air speed of 600 km/h and is heading at 40° and if winds of 120 km/h are blowing in the direction N 30° W. (N 30° W is read as 30° west of north.)

3. A plane has an airspeed of 650 km/h and is headed 30° east of north. The wind is blowing from the south, which causes the actual course of the plane to be 27° east of north.
Find the velocity of the wind and the velocity of the plane with respect to the ground.

4. Three coplanar vectors $\vec{a}$, $\vec{b}$, and $\vec{r}$ make angles of 30°, 120°, and 240° respectively with the horizontal. If $|\vec{a}| = 2$, $|\vec{b}| = 3$, and $|\vec{r}| = 6$, then answer the following.

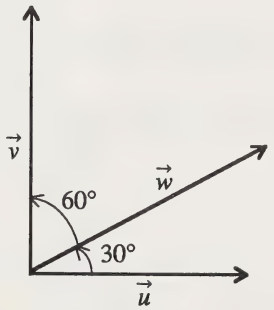
   a. Express $\vec{r}$ as a linear combination of $\vec{a}$ and $\vec{b}$.

   b. Express $\vec{a}$ as a linear combination of $\vec{b}$ and $\vec{r}$.

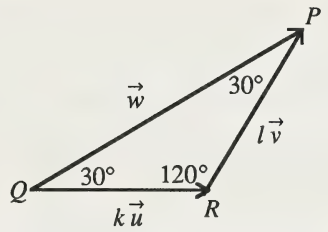
   c. Express $\vec{b}$ as a linear combination of $\vec{a}$ and $\vec{r}$.

5. Two vectors $\vec{u}$ and $\vec{v}$ have a common initial point with an angle of 40° between the two arms.
If $|\vec{u}| = 10$ and $|\vec{v}| = 14$, then find $|\vec{u} + \vec{v}|$ and the angle between $\vec{u} + \vec{v}$ and $\vec{v}$.

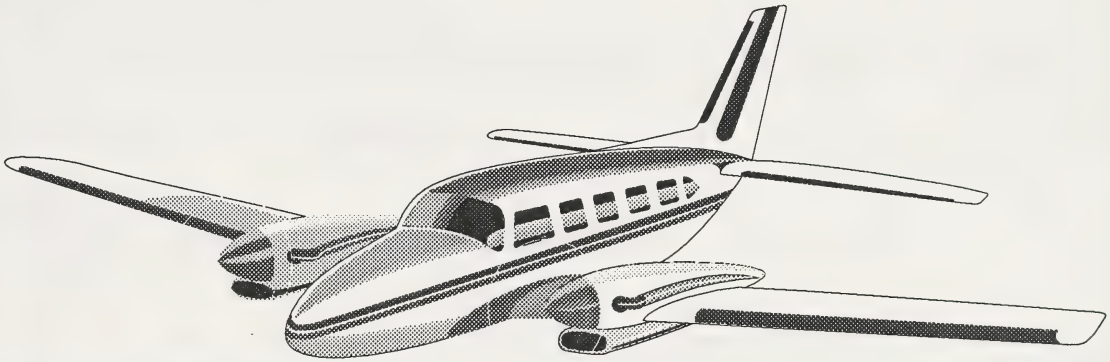
6. In the following diagram $\vec{u}$, $\vec{v}$, and $\vec{w}$ represent three coplanar vectors. If $|\vec{u}| = 10$, $|\vec{v}| = 12$, and $|\vec{w}| = 12$, express $\vec{w}$ as a linear combination of $\vec{u}$ and $\vec{v}$. Also, express $\vec{v}$ as a linear combination of $\vec{u}$ and $\vec{w}$.



7. In the following diagram $\overrightarrow{QP} = \vec{w}$, $\overrightarrow{QR} = k\vec{u}$, and $\overrightarrow{RP} = l\vec{v}$ with $k, l > 0$. If $|\vec{u}| = 2$, $|\vec{v}| = 4$, and $|\vec{w}| = 10$, then express $\vec{w}$ as a linear combination of $\vec{u}$ and $\vec{v}$. (Hint: Use the sine law to solve for k and l.)



For solutions to **Extensions**, turn to the **Appendix, Topic 3**.



Unit Summary

What You Have Learned

Having completed this unit, you should be able to

- define and illustrate skew lines, dihedral angles, geometric vectors, equivalent vectors, collinear vectors, coplanar vectors, vector addition, inverse vectors, vector subtraction, resultant, and equilibrant
- solve problems in three dimensions
- apply the properties of vector addition to solve problems
- apply the properties of multiplication by a scalar to solve problems
- resolve vectors into rectangular components
- express a vector as a linear combination of two other vectors if they are coplanar
- apply your understanding of these concepts to other problem situations

You are now ready to complete the **Unit Assignment.**

Appendix

Solutions

Review

Topic 1 Introduction to Three-Dimensional Vectors

Topic 2 Vector Properties

Topic 3 Vector Applications

Review

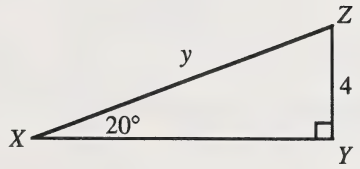
1. a. $c^2 = a^2 + b^2$
$= 3^2 + 4^2$
$= 9 + 16$
$= 25$
$\therefore c = 5$

b. i. $\sin A = \frac{3}{5}$ ii. $\sin B = \frac{4}{5}$

iii. $\cos A = \frac{4}{5}$ iv. $\cos B = \frac{3}{5}$

v. $\tan A = \frac{3}{4}$ vi. $\tan B = \frac{4}{3}$

2.



(Always draw a diagram and label it with the given information.)

To find the sine of 20° on your calculator, enter 20 and press [sin]. Use four decimal places for trigonometric functions.

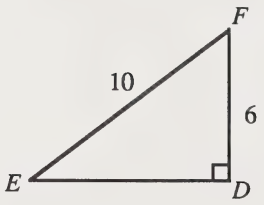
$\sin 20° = \frac{4}{y}$

$y = \frac{4}{\sin 20°}$

$= \frac{4}{0.3420}$

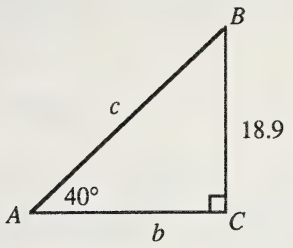
$\doteq 11.7$

3. $\cos F = \frac{6}{10}$
$= 0.6$
$\therefore \angle F \doteq 53°$



To find this angle on your calculator, enter 0.6 and press [Inv] [cos] or enter 0.6 and press [2nd] [cos].

4. You must find $\angle B$, b, and c.

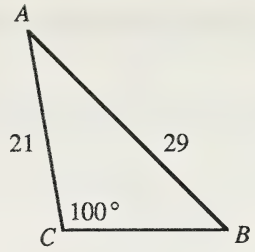


$\angle B = 90° - 40° = 50°$

$$\tan 50° = \frac{b}{18.9}$$
$$b = 18.9(\tan 50°)$$
$$= 18.9(1.1918)$$
$$\doteq 22.5$$

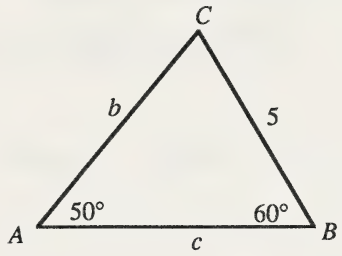
$$c^2 = a^2 + b^2$$
$$= 18.9^2 + 22.5^2$$
$$= 357.21 + 506.25$$
$$= 863.46$$
$$\therefore c \doteq 29.4$$

5.



$$\frac{\sin B}{b} = \frac{\sin C}{c}$$
$$\frac{\sin B}{21} = \frac{\sin 100°}{29}$$
$$\sin B = \frac{21(\sin 100°)}{29}$$
$$= \frac{21(0.9848)}{29}$$
$$= 0.7131$$
$$\therefore \angle B \doteq 45° \text{ (to the nearest degree)}$$

6. You must find $\angle C$, c, and b.



$$\angle C + 50° + 60° = 180°$$
$$\angle C + 110° = 180°$$
$$\angle C = 70°$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\frac{c}{\sin 70°} = \frac{5}{\sin 50°}$$

$$c = \frac{5(\sin 70°)}{\sin 50°}$$

$$= \frac{5(0.9397)}{0.7660}$$

$$\doteq 6.1 \quad \text{(to one decimal place)}$$

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

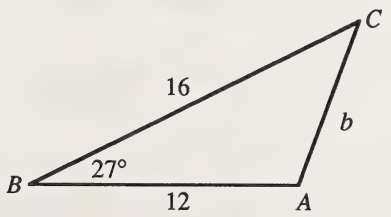
$$\frac{b}{\sin 60°} = \frac{5}{\sin 50°}$$

$$b = \frac{5(\sin 60°)}{\sin 50°}$$

$$= \frac{5(0.8660)}{0.7660}$$

$$\doteq 5.7 \quad \text{(to one decimal place)}$$

7.



$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$= 16^2 + 12^2 - 2(16)(12)\cos 27°$$

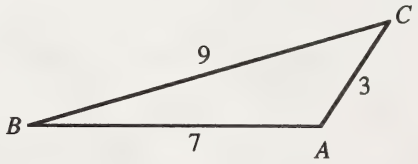
$$= 256 + 144 - 384(0.8910)$$

$$= 400 - 342.144$$

$$= 57.856$$

$$\therefore b \doteq 7.6$$

8.



(The largest angle will be $\angle A$ since it is across from the longest side.)

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$9^2 = 3^2 + 7^2 - 2(3)(7)\cos A$$

$$81 = 9 + 49 - 42\cos A$$

$$81 = 58 - 42\cos A$$

$$23 = -42\cos A$$

$$\cos A = \frac{23}{-42}$$

$$= -0.5476$$

$$\therefore \angle A \doteq 123°$$

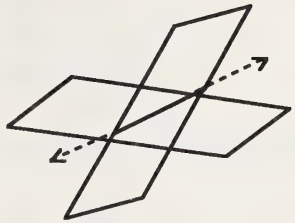
Therefore, $\angle A$ is obtuse since $\cos A$ is negative.

Exploring Topic 1

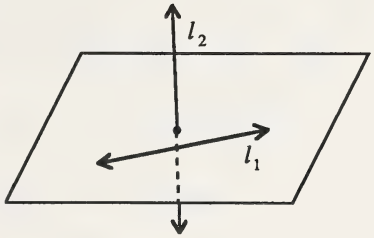
Activity 1

Examine three-dimensional geometric relationships.

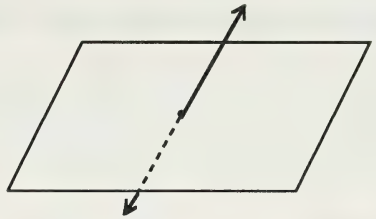
1. a.



b.



c.

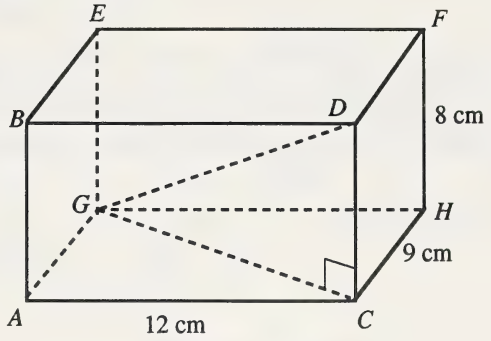


2. a. An infinite amount of planes can be drawn through a given line in space.

   b. Only one plane can be drawn through two intersecting lines.

   c. No plane can pass through two skew lines. By definition two skew lines exist in two different planes.

3. a. Examine right ΔACG.



$$CG^2 = AG^2 + AC^2$$
$$= 9^2 + 12^2$$
$$= 81 + 144$$
$$= 225$$
$$\therefore CG = 15$$

Now observe ΔCDG.

$$GD^2 = CD^2 + GC^2$$
$$= 8^2 + 15^2$$
$$= 64 + 225$$
$$= 289$$
$$\therefore GD = \sqrt{289}$$
$$= 17$$

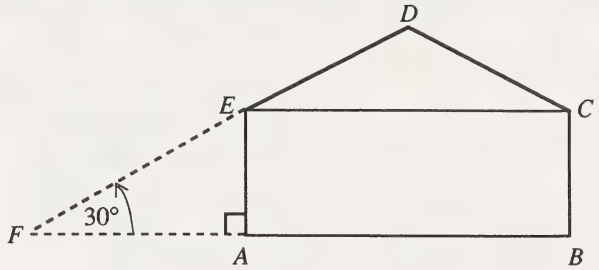
The measure of diagonal GD is 17 cm.

b. Examine $\angle DGC$.

$$\tan \angle DGC = \frac{8}{15}$$
$$= 0.5333$$
$$\therefore \angle DGC \doteq 28°$$

The measure of $\angle DGC$ is approximately 28°.

4. Make a sketch.



The acute dihedral angle you want is $\angle FEA$. The angle between the plane of the roof and the plane of the floor is 30°. The angle between the plane of the wall and the plane of the floor is 90°. Hence, ΔFEA is a right triangle.

$$\angle FEA = 180° - (90° + 30°)$$
$$= 180° - 120°$$
$$= 60°$$

The measure of the acute dihedral angle is 60°.

5. a. Consider the right triangle OPU.

$$PU^2 = OU^2 + OP^2 \quad \text{(by the Pythagorean theorem)}$$
$$= 10^2 + 16^2$$
$$= 100 + 256$$
$$= 356$$
$$\therefore PU = \sqrt{356}$$
$$\doteq 18.87$$

ΔPUV is also a right triangle.

$$\tan \angle PUV = \frac{12}{18.87}$$
$$= 0.6359$$
$$\therefore \angle PUV \doteq 32°$$

The measure of angle PUV is approximately 32°.

b. Consider ΔPUR.

$$PU^2 = 356$$

$$RU^2 = 12^2 + 16^2$$
$$= 144 + 256$$
$$= 400$$

$$PR^2 = 12^2 + 10^2$$
$$= 244$$

$$\cos \angle PUR = \frac{PU^2 + RU^2 - RP^2}{2(PU)(RU)} \quad \text{(by the cosine law)}$$
$$= \frac{356 + 400 - 244}{2(18.87)(20)}$$
$$= \frac{512}{754.8}$$
$$= 0.6783$$
$$\therefore \angle PUR \doteq 47°$$

The measure of angle PUR is 47°.

Activity 2

Examine an introduction to vectors.

1. a. Geometric vectors are vectors that are represented by directed line segments drawn to convey the magnitudes of given quantities.

   b. Equivalent vectors are those that have the same magnitude and direction.

   c. Two vectors are collinear if they lie in the same line or along parallel lines.

   d. Two or more vectors are said to be coplanar if they are contained in the same plane or if they can be moved by sliding to one plane.

2. a. $\overrightarrow{HG}$ and $\overrightarrow{LK}$ are a pair of equivalent vectors.

   b. $\overrightarrow{CD}$ and $\overrightarrow{AB}$ are collinear with $\overrightarrow{EF}$.

   c. $\left|\overrightarrow{EF}\right|$ is not equivalent to $\left|\overrightarrow{IJ}\right|$.

   d. $\overrightarrow{LK}$ has the same magnitude as $\overrightarrow{HG}$.

3. a. $\overrightarrow{BC}$ is equivalent to $\overrightarrow{AD}$.

   b. $\overrightarrow{BA}$ is equivalent to $\overrightarrow{CD}$.

   c. There are no vectors in the parallelogram that are equivalent to $\overrightarrow{DE}$.

4. a. The vectors equivalent to $\overrightarrow{GH}$ are $\overrightarrow{AB}$, $\overrightarrow{DC}$, and $\overrightarrow{FE}$.

   b. $\overrightarrow{HF}$ is a vector equivalent to $\overrightarrow{BD}$.

   c. $\overrightarrow{BC}$ is collinear to $\overrightarrow{AD}$, $\overrightarrow{HE}$, and $\overrightarrow{GF}$.

   d. $\left|\overrightarrow{HF}\right|$ is equal to $\left|\overrightarrow{DB}\right|$ because their magnitudes are the same $\left(\sqrt{164}\text{ units}\right)$.

   e. $\left|\overrightarrow{GC}\right|$ is equal to $\left|\overrightarrow{BF}\right|$ because their magnitudes are the same $\left(\sqrt{200}\text{ units}\right)$.

   f. $\overrightarrow{DC}$ could represent $\overrightarrow{FE}$ because they belong to the same equivalence class, and $\overrightarrow{AD}$ and $\overrightarrow{DC}$ are coplanar; therefore, $\overrightarrow{AD}$ and $\overrightarrow{FE}$ are considered coplanar. (Note that any two vectors maybe considered coplanar.)

Activity 3

Apply vector operations.

1. a. $\overrightarrow{XZ}$

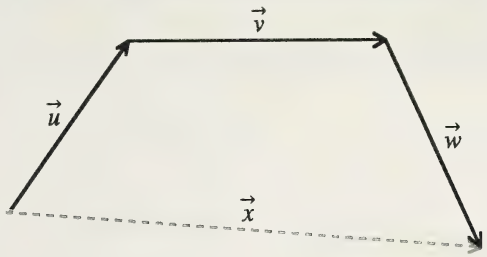
   b. $\overrightarrow{WX}$

   c. $\overrightarrow{WZ}$

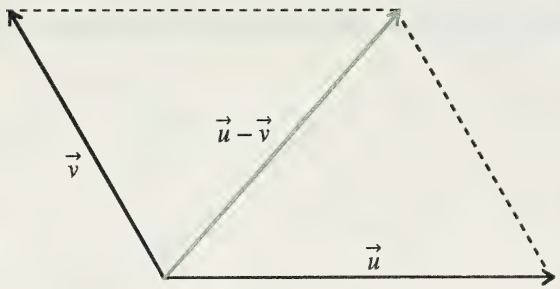
2. $\overrightarrow{MN} = \vec{u}$ and $\overrightarrow{NO} = \vec{v}$

$$\therefore \vec{u} + \vec{v} = \overrightarrow{MN} + \overrightarrow{NO}$$
$$= \overrightarrow{MO}$$

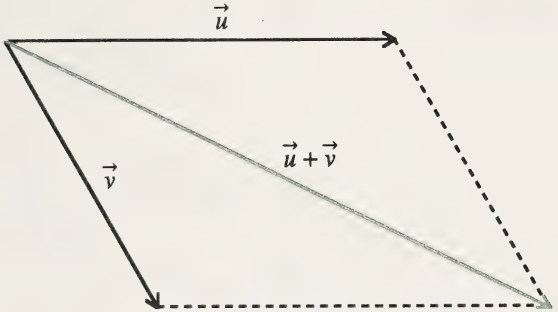
3. $\vec{x} = \vec{u} + \vec{v} + \vec{w}$

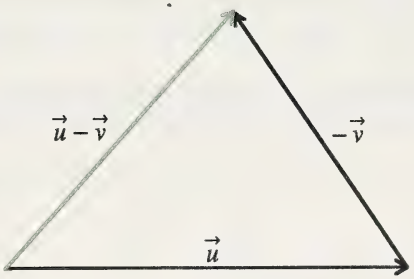


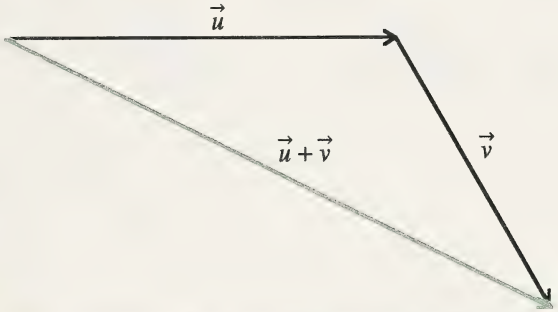
b.



4. a.





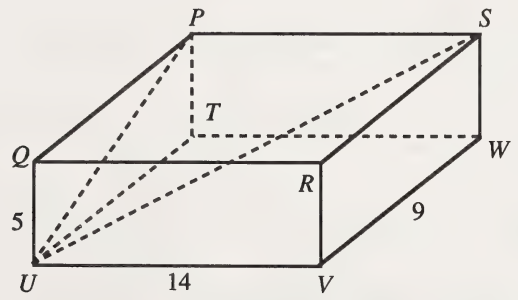


5. a. $\overrightarrow{UW}$

b. $\overrightarrow{UW}$

c. $\overrightarrow{UW}$

6.



a. $\left|\left(\overrightarrow{UQ}+\overrightarrow{UT}\right)+\overrightarrow{UV}\right| = \left|\overrightarrow{UP}+\overrightarrow{UV}\right|$

$= \left|\overrightarrow{US}\right|$

$\left|\overrightarrow{US}\right|^2 = \left|\overrightarrow{UW}\right|^2 + \left|\overrightarrow{WS}\right|^2$

$= \left|\overrightarrow{UV}\right|^2 + \left|\overrightarrow{VW}\right|^2 + \left|\overrightarrow{WS}\right|^2$

$= (14)^2 + (9)^2 + (5)^2$

$= 196 + 81 + 25$

$= 302$

$\left|\overrightarrow{US}\right| \doteq 17$

$\therefore \left|\left(\overrightarrow{UQ}+\overrightarrow{UT}\right)+\overrightarrow{UV}\right| \doteq 17$

b. $\sin \angle SUW = \dfrac{\left|\overrightarrow{SW}\right|}{\left|\overrightarrow{US}\right|}$

$= \dfrac{5}{17}$

$= 0.2941$

$\therefore \angle SUW \doteq 17°$

Extra Help

1. They have the same magnitude (length), but they are opposite in direction. In other words one vector is the inverse of the other.

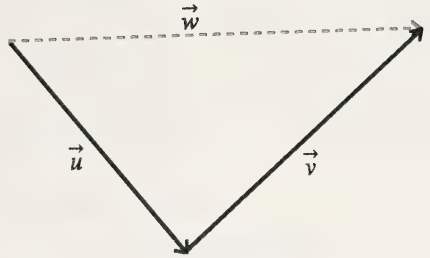
2. a. $\vec{e}$ is the inverse of $\vec{f}$.

   b. $\vec{i}$ is the inverse of $\vec{j}$.

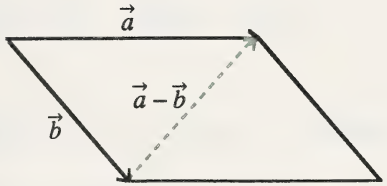
   c. $\vec{o}$ is the inverse of $\vec{q}$.

   d. $\vec{p}$ is the inverse of $\vec{r}$.

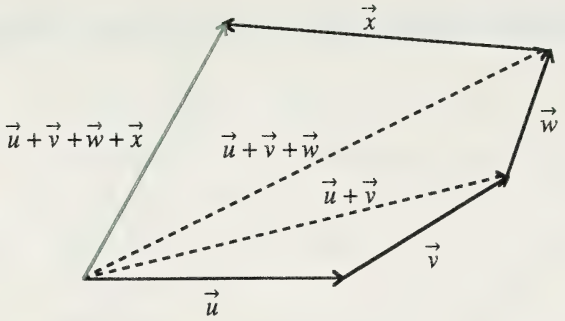
3. $\vec{w} = \vec{u} + \vec{v}$



4.



5.

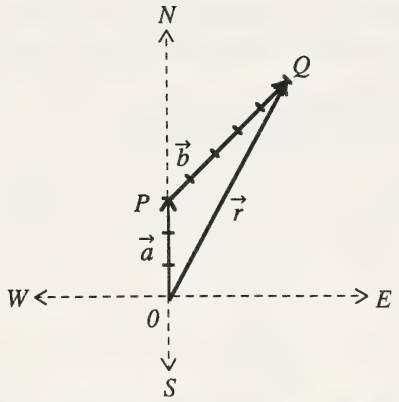


6. The vector quantities are weight, velocity, momentum, displacement, force, and magnetic field.

   The scalar quantities are mass, calorie, specific heat, energy, distance, speed, density, volume, and real numbers.

Extensions

1. a.



Scale: 1 cm ≐ 2 km

From the graph $\vec{r} = \vec{a} + \vec{b}$, the magnitude is measured to be 6.5 nautical miles, and the direction is 71° north of east.

b. The cosine law is used to calculate the magnitude of the resultant.

$$\left|\vec{r}\right|^2 = \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 - 2\left|\vec{a}\right|\left|\vec{b}\right|\cos \angle OPQ$$

$$= 4^2 + 3^2 - 2(4)(3)\cos 135°$$

$$= 25 - 24(-\cos 45°)$$

$$= 25 - 24\left(-\frac{1}{\sqrt{2}}\right)$$

$$= 25 + 24\left(\frac{\sqrt{2}}{2}\right)$$

$$= 25 + 12\sqrt{2}$$

$$= 16.97 + 25$$

$$= 41.97$$

$$\therefore \left|\vec{r}\right| \doteq 6.48$$

The sine law is us find the direction of the resultant.

$$\frac{\left|\vec{b}\right|}{\sin \angle POQ} = \frac{\left|\vec{r}\right|}{\sin \angle OPQ}$$

$$\frac{3}{\sin \angle POQ} = \frac{6.48}{\sin 135°}$$

$$\sin \angle POQ = \frac{3 \sin 135°}{6.48}$$

$$= \frac{3 \sin 45°}{6.48}$$

$$= \frac{3}{\sqrt{2}(6.48)}$$

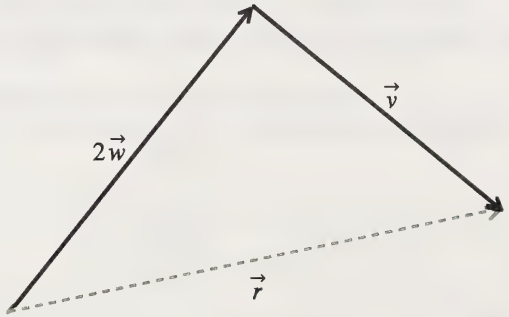
$$= \frac{3}{9.16}$$

$$= 0.3274$$

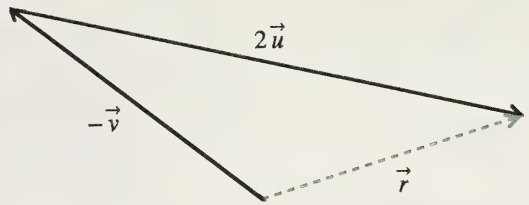
$$\therefore \angle POQ \doteq 19° \quad (\text{Direction} = 90° - 19° = 71°)$$

Therefore, the magnitude of the resultant displacement is 6.48 km and the direction is 71° north of east.

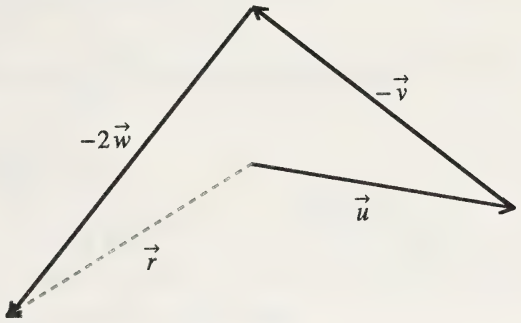
2. a. $\vec{r} = \vec{v} + 2\vec{w}$



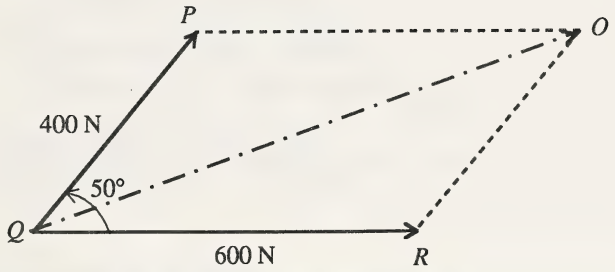
b. $\vec{r} = 2\vec{u} - \vec{v}$



c. $\vec{r} = \vec{u} - \vec{v} - 2\vec{w}$



3.



$$\left|\overrightarrow{QO}\right|^2 = \left|\overrightarrow{QR}\right|^2 + \left|\overrightarrow{RO}\right|^2 - 2\left|\overrightarrow{QR}\right|\left|\overrightarrow{RO}\right| \cos 130°$$

$$= (600)^2 + (400)^2 - 2(600)(400)\cos 130°$$

$$= 360\,000 + 160\,000 - 480\,000(-\cos 50°)$$

$$= 520\,000 + 480\,000 \cos 50°$$

$$= 520\,000 + 480\,000(0.6428)$$

$$= 520\,000 + 308\,538$$

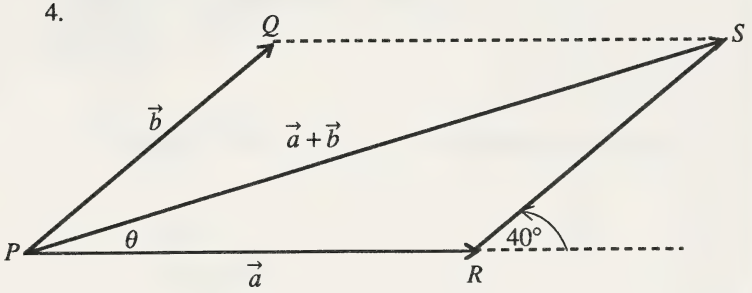
$$= 828\,538$$

$$\therefore \left|\overrightarrow{QO}\right| = \sqrt{828\,538}$$

$$\doteq 910$$

The mule is experiencing a force of approximately 910 N.

4.



In ΔPRS $\left|\overrightarrow{PR}\right| = 14$, $\left|\overrightarrow{RS}\right| = 10$, and $\angle PRS = 140°$.

$$\left|\overrightarrow{PS}\right|^2 = \left|\overrightarrow{PR}\right|^2 + \left|\overrightarrow{RS}\right|^2 - 2\left|\overrightarrow{PR}\right|\left|\overrightarrow{RS}\right|\cos\angle PRS$$
$$= (14)^2 + (10)^2 - (2)(14)(10)\cos 140°$$
$$= 196 + 100 - 280(-\cos 40°)$$
$$= 196 + 100 + 280(\cos 40°)$$
$$= 196 + 100 + 280(0.7660)$$
$$= 296 + 214.49$$
$$= 510.49$$
$$\therefore \left|\overrightarrow{PS}\right| \doteq 22.6$$

Therefore, $\left|\vec{a} + \vec{b}\right|$ is approximately 22.6 units.

In order to find θ, use the sine law.

$$\frac{\left|\overrightarrow{SR}\right|}{\sin\theta} = \frac{\left|\overrightarrow{PS}\right|}{\sin\angle PRS}$$
$$\frac{10}{\sin\theta} = \frac{22.6}{\sin 140°}$$
$$\sin\theta = \frac{10\sin 140°}{22.6}$$
$$\sin\theta = \frac{10\sin 40°}{22.6}$$
$$= \frac{10(0.6428)}{22.6}$$
$$= \frac{6.4279}{22.6}$$
$$= 0.2844$$
$$\therefore \theta \doteq 17°$$

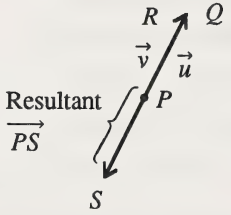
The angle between $\left(\vec{a} + \vec{b}\right)$ and $\vec{a}$ is approximately 17°.

Exploring Topic 2

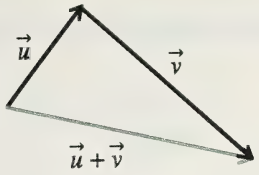
Activity 1

Define and illustrate vector properties for addition and subtraction.

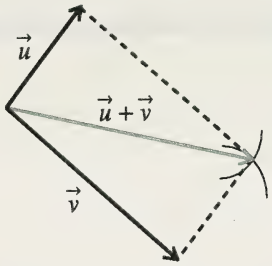
1. a.



b.

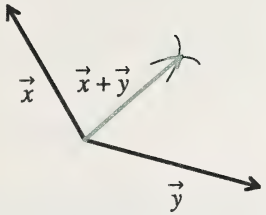


(triangle method)

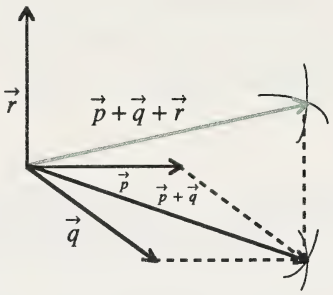


(parallelogram method)

2. a.



b.



3. a. $\overrightarrow{MO}$

b. $\overrightarrow{MO}$

c. $\overrightarrow{ON}$

d. $\overrightarrow{MP}+\overrightarrow{OP}=\overrightarrow{NO}+\overrightarrow{OP}=\overrightarrow{NP}$

e. $\overrightarrow{MP}$

f. $\overrightarrow{PN}$

4. a. $\overrightarrow{UW}$

b. $\overrightarrow{UX}+\overrightarrow{XW}=\overrightarrow{UW}$

c. $\overrightarrow{UY}+\overrightarrow{YW}=\overrightarrow{UW}$

5. a. $\overrightarrow{AC}$

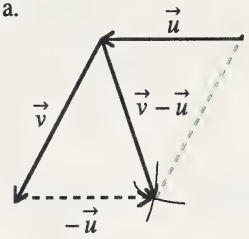
b. $\overrightarrow{AC}$

c. $\vec{u}+\vec{v}$ and $\vec{v}+\vec{u}$ are equivalent.

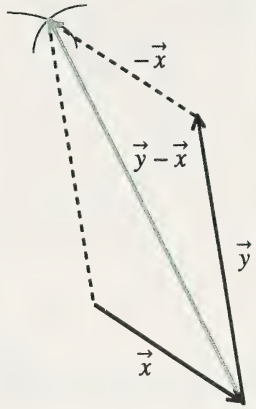
d. $\overrightarrow{DB}$

e. $\overrightarrow{BD}$

6. a.



b.



7. a. $\left(\vec{u}+\vec{v}\right)+\left(-\vec{u}\right)$

$=\left(\vec{v}+\vec{u}\right)+\left(-\vec{u}\right)$ (by the commutative property)

$=\vec{v}+\left[\vec{u}+\left(-\vec{u}\right)\right]$ (by the associative property)

$=\vec{v}+\vec{0}$ (by the inverse property)

$=\vec{v}$ (by the additive identity property)

b. $\vec{u}+\vec{v}+\left(-\vec{u}\right)+\vec{w}+\left(-\vec{v}\right)$

$=\vec{u}+\left(-\vec{u}\right)+\vec{v}+\left(-\vec{v}\right)+\vec{w}$ (by the commutative property)

$=\left[\vec{u}+\left(-\vec{u}\right)\right]+\left[\vec{v}+\left(-\vec{v}\right)\right]+\vec{w}$ (by the associative property)

$=\vec{0}+\vec{0}+\vec{w}$ (by the inverse property)

$=\vec{w}$ (by the additive identity property)

8. In ΔABC the following may be stated using the cosine law:

$$\left|\overrightarrow{AC}\right|^2=\left|\overrightarrow{AD}\right|^2+\left|\overrightarrow{DC}\right|^2-2\left|\overrightarrow{AD}\right|\left|\overrightarrow{DC}\right|\cos\angle CDA$$

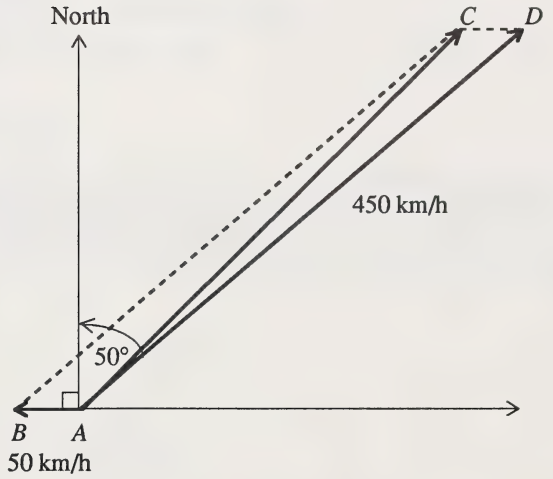
$$=(450)^2+(50)^2-2(450)(50)\cos 40°$$

$$=202\,500+2500-45\,000(0.7660)$$

$$=205\,000-34\,472$$

$$=170\,528$$

$$\therefore\left|\overrightarrow{AC}\right|\doteq 413$$



$$\frac{\left|\overrightarrow{DC}\right|}{\sin \angle CAD} = \frac{\left|\overrightarrow{AC}\right|}{\sin \angle CDA}$$

$$\frac{50}{\sin \angle CAD} = \frac{413}{\sin 40°}$$

$$\sin \angle CAD = \frac{50 \sin 40°}{413}$$

$$= \frac{50(0.6428)}{413}$$

$$= 0.0778$$

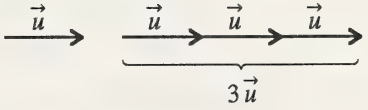
$$\therefore \angle CAD \doteq 4° \quad (\text{Direction} = 50° - 4° = 46°)$$

Therefore, the ground speed is 413 km/h in a direction 46° from north.

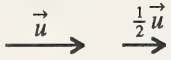
Activity 2

Define and use the properties of multiplication by a scalar in various problem situations.

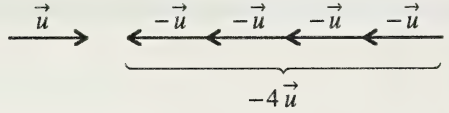
1. a. $3\vec{u}$



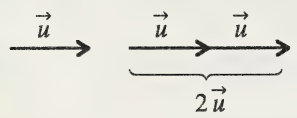
b. $\frac{1}{2}\vec{u}$



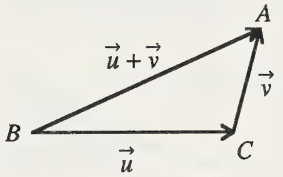
c. $-4\vec{u}$

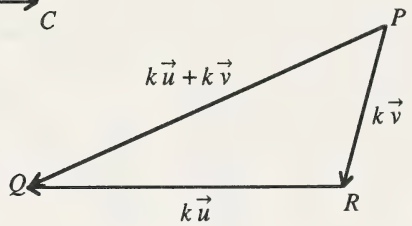


d. $2\vec{u}$



2. • $k < 0$





Let $\overrightarrow{BC} = \vec{u}$ and $\overrightarrow{CA} = \vec{v}$; then, $\overrightarrow{BA} = \vec{u} + \vec{v}$.

Construct ΔPQR with $\overrightarrow{PQ} \parallel \overrightarrow{AB}$, $\overrightarrow{QR} \parallel \overrightarrow{BC}$, $\overrightarrow{RP} \parallel \overrightarrow{CA}$, and $\overrightarrow{RQ} = k\vec{u}$.

Therefore, $\Delta PQR \sim \Delta ABC$.

$\therefore \overrightarrow{PQ} = k\overrightarrow{BA} = k\left(\vec{u}+\vec{v}\right)$ (1)

$= \overrightarrow{PR} + \overrightarrow{RQ}$

$= k\vec{v} + k\vec{u}$ (2)

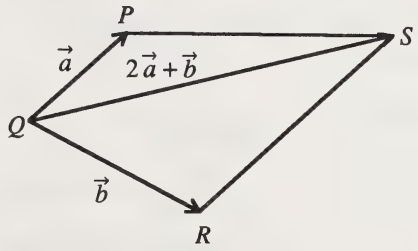
Therefore, by combining (1) and (2) you may state that $k\left(\vec{u}+\vec{v}\right) = k\vec{u} + k\vec{v}$.

- $k = 0$

When $k = 0$, then $k\left(\vec{u}+\vec{v}\right) = \vec{0}$, $k\vec{u} = \vec{0}$, and $k\vec{v} = \vec{0}$.

Therefore, $k\vec{u} + k\vec{v} = \vec{0} + \vec{0} = \vec{0}$ and $k\left(\vec{u}+\vec{v}\right) = k\vec{u} + k\vec{v}$.

3.



$\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QS}$

$= -\overrightarrow{QP} + \overrightarrow{QS}$

$= -\vec{a} + \left(2\vec{a}+\vec{b}\right)$

$= \vec{a} + \vec{b}$

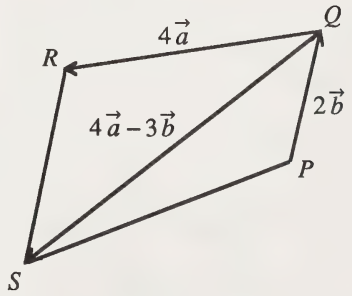
$\overrightarrow{RS} = \overrightarrow{RQ} + \overrightarrow{QS}$

$= -\overrightarrow{QR} + \overrightarrow{QS}$

$= -\vec{b} + \left(2\vec{a}+\vec{b}\right)$

$= 2\vec{a}$

4.



$\overrightarrow{RS} = \overrightarrow{RQ} + \overrightarrow{QS}$

$= -\overrightarrow{QR} + \overrightarrow{QS}$

$= -4\vec{a} + \left(4\vec{a}-3\vec{b}\right)$

$= -3\vec{b}$

$\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QS}$

$= 2\vec{b} + \left(4\vec{a}-3\vec{b}\right)$

$= 4\vec{a} - \vec{b}$

5. a. $\overrightarrow{BC} + \overrightarrow{BD} + \overrightarrow{BE} = 2\vec{p} + 4\vec{q} + 3\vec{r}$

b. $\left|\overrightarrow{BC} + \overrightarrow{BD} + \overrightarrow{BE}\right| = \left|\overrightarrow{BG}\right|$

$$= \sqrt{\left|\overrightarrow{BE}\right|^2 + \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{BD}\right|^2}$$

$$= \sqrt{6^2 + 2^2 + 12^2}$$

$$= \sqrt{184}$$

$$= 2\sqrt{46}$$

6. Proof:

R, S, and T are collinear with S being the midpoint of segment RT.

$\therefore \overrightarrow{RT} = \overrightarrow{RS} + \overrightarrow{ST}$ (vector addition)

$= \overrightarrow{RS} + \overrightarrow{RS}$ (since $\overrightarrow{RS} = \overrightarrow{ST}$)

$= 2\overrightarrow{RS}$ (1) (scalar multiplication)

From the diagram the following may be stated:

$\overrightarrow{RT} = \overrightarrow{RO} + \overrightarrow{OT}$ (2)

$\overrightarrow{RS} = \overrightarrow{RO} + \overrightarrow{OS}$ (3)

Substitute (2) and (3) in equation (1).

$$\overrightarrow{RO} + \overrightarrow{OT} = 2\left(\overrightarrow{RO} + \overrightarrow{OS}\right)$$

$\overrightarrow{RO} + \overrightarrow{OT} = 2\overrightarrow{RO} + 2\overrightarrow{OS}$ (distributive property)

$\overrightarrow{RO} - 2\overrightarrow{RO} + \overrightarrow{OT} = 2\overrightarrow{OS}$ (subtracting $2\overrightarrow{RO}$ from each side)

$-\overrightarrow{RO} + \overrightarrow{OT} = 2\overrightarrow{OS}$ (vector subtraction)

$\overrightarrow{OR} + \overrightarrow{OT} = 2\overrightarrow{OS}$ (inverse vector)

Therefore, $\overrightarrow{OR} + \overrightarrow{OT} = 2\overrightarrow{OS}$.

Extra Help

1. a. $\overrightarrow{BD}$
   b. $\overrightarrow{BE}$
   c. $\overrightarrow{HC}$
   d. $\overrightarrow{HC}$
   e. $\overrightarrow{BE}$
   f. $\overrightarrow{HB}$
   g. $\overrightarrow{BD}$
   h. $\overrightarrow{HC}$
   i. $\vec{0}$
   j. $\vec{0}$
   k. $\vec{0}$
   l. $\vec{0}$
   m. $\vec{0}$

2. Let the magnitude of each of the forces be x.

In ΔCAD $\left|\overrightarrow{AC}\right|^2 = \left|\overrightarrow{AD}\right|^2 + \left|\overrightarrow{DC}\right|^2$.

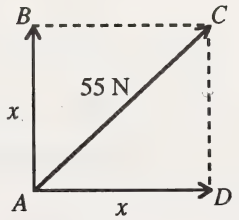
$$\therefore \left|\overrightarrow{AC}\right|^2 = 2x^2$$

$$2x^2 = (55)^2$$

$$2x^2 = 3025$$

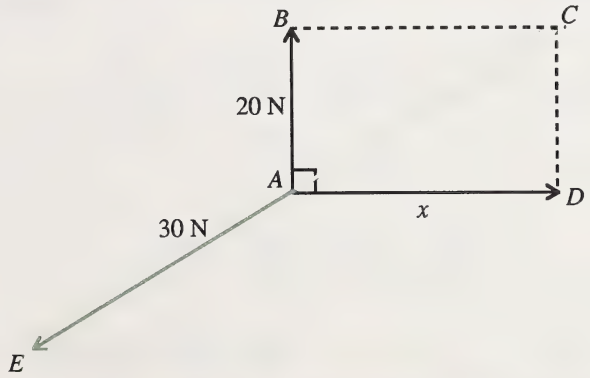
$$x^2 = 1512.5$$

$$x \doteq 38.9$$



The magnitude of each force is approximately 38.9 N.

3. Since the equilibrant of the 20 N and x forces is 30 N, the resultant force is also a force of 30 N; thus, $\overrightarrow{AC}$ is 30 N.



The resultant force $\overrightarrow{AC}$ of the 20 N and x forces must also be 30 N since the equilibrant force is 30 N.

$$\left|\overrightarrow{AD}\right|^2 + \left|\overrightarrow{DC}\right|^2 = \left|\overrightarrow{AC}\right|^2$$

$$\left|\vec{x}\right|^2 + (20)^2 = (30)^2$$

$$\left|\vec{x}\right|^2 + 400 = 900$$

$$\left|\vec{x}\right|^2 = 900 - 400$$

$$= 500$$

$$\therefore \left|\vec{x}\right| \doteq 22.4$$

$$\sin \angle CAD = \frac{\left|\overrightarrow{DC}\right|}{\left|\overrightarrow{AC}\right|}$$

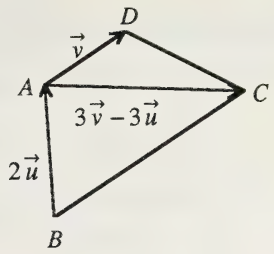
$$= \frac{20}{30}$$

$$= 0.6667$$

$$\therefore \angle CAD \doteq 42^\circ \quad [\angle DAE = -(180 - \angle CAD) \doteq -138^\circ]$$

Thus, the value of the horizontal force is approximately 22.4 N and the angle of the 30 N equilibrant force is approximately -138° with respect to the horizontal.

4.



$$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$$
$$= 2\vec{u} + \left(3\vec{v} - 3\vec{u}\right)$$
$$= 3\vec{v} - \vec{u}$$

$$\overrightarrow{DC} = \overrightarrow{DA} + \overrightarrow{AC}$$
$$= -\vec{v} + \left(3\vec{v} - 3\vec{u}\right)$$
$$= 2\vec{v} - 3\vec{u}$$

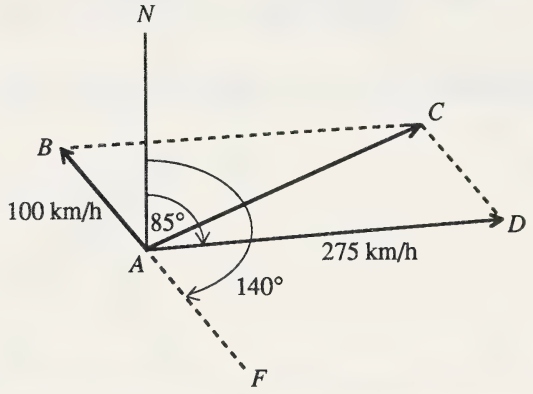
5. If $\overrightarrow{BA} = \vec{u}$ and $\overrightarrow{BC} = \vec{v}$, then the following may be stated:

$$\vec{u} + \vec{v} = \overrightarrow{BA} + \overrightarrow{BC}$$
$$= \overrightarrow{BA} + \overrightarrow{AD} \quad \text{(Opposites sides of a parallelogram are parallel.)}$$
$$= \overrightarrow{BD}$$

$$\vec{u} - \vec{v} = \overrightarrow{BA} - \overrightarrow{BC}$$
$$= \overrightarrow{BA} + \left(-\overrightarrow{BC}\right)$$
$$= \overrightarrow{CD} + \overrightarrow{DA}$$
$$= \overrightarrow{CA}$$

$$\vec{v} - \vec{u} = \overrightarrow{BC} - \overrightarrow{BA}$$
$$= \overrightarrow{BC} + \left(-\overrightarrow{BA}\right)$$
$$= \overrightarrow{AD} + \overrightarrow{DC}$$
$$= \overrightarrow{AC}$$

6.



In ΔACD, $\angle CDA$ is $55°(140°-85°=55°)$, since $\angle CDA = \angle DAF$ (alternate angles).

$$\left|\overrightarrow{AC}\right|^2 = \left|\overrightarrow{AD}\right|^2 + \left|\overrightarrow{DC}\right|^2 - 2\left|\overrightarrow{AD}\right|\left|\overrightarrow{DC}\right| \cos \angle CDA$$

$$= (275)^2 + (100)^2 - 2(275)(100)\cos 55°$$

$$= 75\,625 + 10\,000 - 55\,000(0.5736)$$

$$= 85\,625 - 31\,546.7$$

$$= 54\,078.3$$

$$\therefore \left|\overrightarrow{AC}\right| \doteq 233$$

$$\frac{\left|\overrightarrow{CD}\right|}{\sin \angle CAD} = \frac{\left|\overrightarrow{AC}\right|}{\sin \angle CDA}$$

$$\sin \angle CAD = \frac{100 \sin 55°}{233}$$

$$= \frac{100(0.8192)}{233}$$

$$= 0.3516$$

$$\therefore \angle CAD \doteq 21°$$

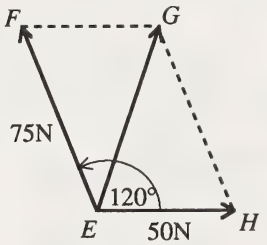
$$\angle CAN \doteq 85° - 21°$$

$$\doteq 64°$$

The ground speed is 233 km/h in a direction 64° from north.

Extensions

1.



From ΔEGH the following may be stated using the cosine law:

$$\left|\overrightarrow{EG}\right|^2 = \left|\overrightarrow{EH}\right|^2 + \left|\overrightarrow{HG}\right|^2 - 2\left|\overrightarrow{EH}\right|\left|\overrightarrow{HG}\right| \cos \angle GHE$$

$$= (50)^2 + (75)^2 - 2(50)(75)\cos 60°$$

$$= 2500 + 5625 - 7500(0.5)$$

$$= 8125 - 3750$$

$$= 4375$$

$$\therefore \left|\overrightarrow{EG}\right| \doteq 66.14$$

From ΔEIJ the following may be stated using the Pythagorean theorem:

$$\left|\overrightarrow{EJ}\right|^2 = \left|\overrightarrow{EI}\right|^2 + \left|\overrightarrow{EG}\right|^2$$

$$= (90)^2 + 4375$$

$$= 12\,475$$

$$\therefore \left|\overrightarrow{EJ}\right| \doteq 111.7$$

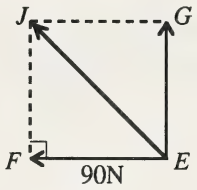
$\overrightarrow{EJ}$ is the resultant of $\overrightarrow{EG}$ and $\overrightarrow{EI}$. Note that $\overrightarrow{EI} \perp \overrightarrow{EG}$ since $\overrightarrow{EG}$ is in the plane formed by $\overrightarrow{EF}$ and $\overrightarrow{EH}$.

$$\sin \angle JEG = \frac{\left|\overrightarrow{JG}\right|}{\left|\overrightarrow{EJ}\right|}$$

$$= \frac{90}{111.7}$$

$$= 0.8058$$

$$\therefore \angle JEG \doteq 54°$$



Therefore, the resultant is a force of 111.7 N acting at an angle of 54° to the plane containing the first two forces.

2. Let $\overrightarrow{R_1}$ be $\vec{r}$.

$$\left|\overrightarrow{R_1}\right| = \left|3\vec{u} - \vec{v} + 4\vec{w}\right|$$

$$= \sqrt{[3(2)]^2 - (3)^2 + [4(1)]^2}$$

$$= \sqrt{(6)^2 - (3)^2 + (4)^2}$$

$$= \sqrt{36 - 9 + 16}$$

$$= \sqrt{43}$$

A unit vector $\overrightarrow{R_2}$ which is parallel to $\overrightarrow{R_1}$ is given by $\dfrac{\overrightarrow{R_1}}{\left|\overrightarrow{R_1}\right|}$.

$$\therefore \overrightarrow{R_2} = \frac{3\vec{u} - \vec{v} + 4\vec{w}}{\sqrt{43}}$$

Verification:

$$\left|\frac{3\vec{u}}{\sqrt{43}} - \frac{\vec{v}}{\sqrt{43}} + \frac{4\vec{w}}{\sqrt{43}}\right|$$

$$= \sqrt{\left(\frac{3(2)}{\sqrt{43}}\right)^2 - \left(\frac{3}{\sqrt{43}}\right)^2 + \left(\frac{4(1)}{\sqrt{43}}\right)^2}$$

$$= \sqrt{\frac{36}{43} - \frac{9}{43} + \frac{16}{43}}$$

$$= \sqrt{\frac{43}{43}}$$

$$= 1$$

3. a. $\left|\vec{b}\right| = \left|2\vec{u} - 4\vec{v} - 3\vec{w}\right|$

$= \sqrt{2^2 + (-4)^2 + (-3)^2}$

$= \sqrt{4 + 16 + 9}$

$= \sqrt{29}$

b.

$\vec{a} + \vec{b} + \vec{c} = \left(3\vec{u} - 2\vec{v} + \vec{w}\right) + \left(2\vec{u} - 4\vec{v} - 3\vec{w}\right) + \left(-\vec{u} + 2\vec{v} + 2\vec{w}\right)$

$= \left(4\vec{u} - 4\vec{v} + 0\vec{w}\right)$

$\therefore \left|\vec{a} + \vec{b} + \vec{c}\right| = \left|4\vec{u} - 4\vec{v} + 0\vec{w}\right|$

$= \sqrt{4^2 + (-4)^2 + 0^2}$

$= \sqrt{16 + 16}$

$= \sqrt{32}$

$= 4\sqrt{2}$

c.

$\vec{a} - 2\vec{b} - 3\vec{c} = \left(3\vec{u} - 2\vec{v} + \vec{w}\right) - 2\left(2\vec{u} - 4\vec{v} - 3\vec{w}\right) - 3\left(-\vec{u} + 2\vec{v} + 2\vec{w}\right)$

$= 3\vec{u} - 2\vec{v} + \vec{w} - 4\vec{u} + 8\vec{v} + 6\vec{w} + 3\vec{u} - 6\vec{v} - 6\vec{w}$

$= 2\vec{u} + 0\vec{v} + \vec{w}$

$\therefore \left|\vec{a} - 2\vec{b} - 3\vec{c}\right| = \left|2\vec{u} + 0\vec{v} + \vec{w}\right|$

$= \sqrt{2^2 + 0^2 + 1^2}$

$= \sqrt{4 + 1}$

$= \sqrt{5}$

Exploring Topic 3

Activity 1

Define and apply resolution of forces.

1. a. $\left|\vec{b}\right| = \left|\vec{F}\right| \cos\theta$

$= (90\text{ N}) \cos 60°$

$= (90\text{ N})(0.5)$

$= 45\text{ N}$

$\left|\vec{a}\right| = \left|\vec{F}\right| \sin\theta$

$= (90\text{ N}) \sin 60°$

$= (90\text{ N}) \frac{\sqrt{3}}{2}$

$= 45\sqrt{3}\text{ N}$

b. $\left|\vec{b}\right| = \left|\vec{F}\right| \cos\theta$

$= (160\text{ N}) \cos 45°$

$= (160\text{ N})\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{160\sqrt{2}}{2}\text{ N}$

$= 80\sqrt{2}\text{ N}$

$\left|\vec{a}\right| = \left|\vec{F}\right| \sin\theta$

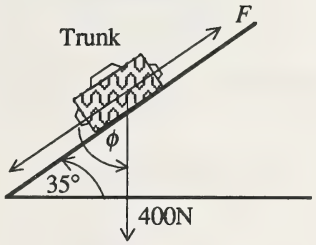
$= (160\text{ N}) \sin 45°$

$= (160\text{ N})\left(\frac{1}{\sqrt{2}}\right)$

$= (160\text{ N})\left(\frac{\sqrt{2}}{2}\right)$

$= 80\sqrt{2}\text{ N}$

2.



(ϕ is the angle between the line parallel to the incline and the vertical.)

$\left|\vec{F}\right| = \left|\vec{W}\right| \cos\phi$

$= 400 \cos 55°$

$= 400(0.5736)$

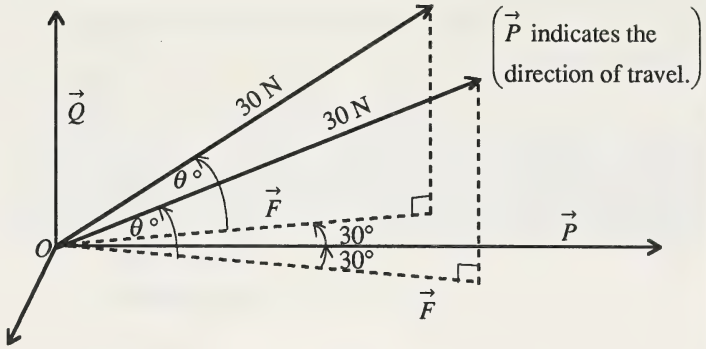
$= 229.4$

$\doteq 229$

The force required to keep the trunk stationary is 229 N.

3. First make a sketch of the situation.

Let the force pulling the sleigh forward be $\vec{P}$, the force tending to lift the sleigh upward be $\vec{Q}$, and the horizontal component of each force be $\vec{F}$.



a. $\left|\vec{F}\right| = 30 \cos\theta$

(Resolve forces into horizontal components.)

$\left|\vec{P}\right| = \left|\vec{F}\right| \cos 30° + \left|\vec{F}\right| \cos 30°$

$= 2\left|\vec{F}\right| \cos 30°$

$= 2(30 \cos\theta) \cos 30°$

However, it is know that $\left|\vec{P}\right|$ is 45 N.

$\therefore 45 = 60 \cos\ \theta\ \cos\ 30°$

$$\cos\ \theta = \frac{45}{60 \cos\ 30°}$$

$$= \left(\frac{45}{60}\right)\left(\frac{2}{\sqrt{3}}\right)$$

$$= \frac{3}{2\sqrt{3}}$$

$$= \frac{\sqrt{3}}{2}$$

$\therefore \theta = 30°$

The forces are exerted at a 30° angle to the horizontal.

b. $\left|\vec{Q}\right| = 30 \sin\ 30° + 30 \sin\ 30°$

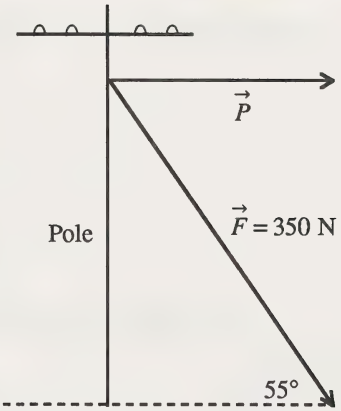
$= 60 \sin\ 30°$

$= 60(0.5)$

$= 30$

A 30 N force is pulling the sleigh upward.

4. Let $\vec{P}$ be the horizontal component of $\vec{F}$.



$\therefore \left|\vec{P}\right| = \left|\vec{F}\right| \cos\ 55°$

$= 350 \cos\ 55°$

$= 350(0.5736)$

$= 200.75$

$\doteq 201$

The force $\vec{P}$ is approximately 201 N.

5. $\left|\overrightarrow{F_V}\right| = \left|\vec{P}\right| \cos\ \theta$

$= 1000(0.9659)$

$= 965.9$

$\doteq 966$

The vertical component is 966 N.

$$\left|\overrightarrow{F_H}\right| = \left|\vec{P}\right| \sin\theta$$

$$= 1000 \sin 15°$$

$$= 1000(0.2588)$$

$$= 258.8$$

$$\doteq 259$$

The horizontal component is 259 N.

6. $\left|\overrightarrow{F_1}\right| = 40 \cos 35°$

$$\left|\overrightarrow{F_2}\right| = 40 \cos 35°$$

$$\left|\vec{P}\right| = \left|\overrightarrow{F_1}\right| \cos 30° + \left|\overrightarrow{F_2}\right| \cos 30°$$

$$= 40 \cos 35° \cos 30° + 40 \cos 35° \cos 30°$$

$$= 80 \cos 35° \cos 30°$$

$$= 80(0.8192)(0.8660)$$

$$= 56.75$$

$$\doteq 57$$

The force that draws the toboggan forward is 57 N.

$$\left|\vec{Q}\right| = 40 \sin 35° + 40 \sin 35°$$

$$= 80 \sin 35°$$

$$= 80(0.5736)$$

$$= 45.89$$

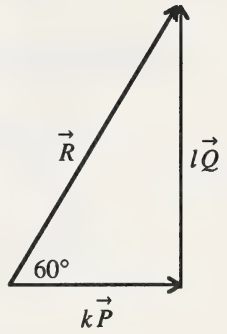
$$\doteq 46$$

The force that tends to lift the toboggan is 46 N.

Activity 2

Define and apply linear combinations of geometric vectors.

1. Draw a triangle to represent the situation.



a. Let $\vec{R} = k\vec{P} + l\vec{Q}$. (1)

$$k\left|\vec{P}\right| = \left|\vec{R}\right| \cos 60°$$

$$10k = 15 \cos 60°$$

$$k = \frac{15}{20}$$

$$= 0.75 \quad (2)$$

$$l\left|\vec{Q}\right| = \left|\vec{R}\right| \sin 60°$$

$$\therefore 12l = 15 \sin 60°$$

$$l = \frac{15\sqrt{3}}{(12)(2)}$$

$$= \frac{5\sqrt{3}}{8} \quad (3)$$

Now substitute (2) and (3) in (1).

$$\therefore \vec{R} = \frac{3}{4}\vec{P} + \frac{5\sqrt{3}}{8}\vec{Q}$$

b. Let $l\vec{Q} = \vec{R} - k\vec{P}$.

$$\frac{5\sqrt{3}}{8}\vec{Q} = \vec{R} - \frac{3}{4}\vec{P}$$

$$\vec{Q} = \frac{8\left(4\vec{R} - 3\vec{P}\right)}{(4)5\sqrt{3}}$$

$$= \frac{8\vec{R} - 6\vec{P}}{5\sqrt{3}}$$

$$= \frac{8\sqrt{3}\vec{R}}{15} - \frac{6\sqrt{3}\vec{P}}{15}$$

2. Let $\vec{w} = k\left|\vec{u}\right| + l\left|\vec{v}\right|$ where $l,\ k \neq 0$.

The following may be stated using the sine law:

$$\frac{\left|\vec{w}\right|}{\sin 120°} = \frac{l\left|\vec{v}\right|}{\sin 30°} \qquad \frac{\left|\vec{w}\right|}{\sin 120°} = \frac{k\left|\vec{u}\right|}{\sin 30°}$$

Now substitute in $\left|\vec{w}\right| = 20$, $\left|\vec{v}\right| = 8$, and $\left|\vec{u}\right| = 4$ to solve for l and k.

(Remember: $\sin 120° = \sin 60°$)

$$\frac{20}{\sin 60°} = \frac{l(8)}{\sin 30°} \qquad \frac{20}{\sin 60°} = \frac{k(4)}{\sin 30°}$$

$$l = \frac{20 \sin 30°}{8 \sin 60°} \qquad k = \frac{20 \sin 30°}{4 \sin 60°}$$

$$= \frac{20\left(\frac{1}{2}\right)}{8\left(\frac{\sqrt{3}}{2}\right)} \qquad = \frac{20\left(\frac{1}{2}\right)}{4\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{10}{4\sqrt{3}} \qquad = \frac{10}{2\sqrt{3}}$$

$$= \frac{10\sqrt{3}}{12} \qquad = \frac{10\sqrt{3}}{6}$$

$$= \frac{5\sqrt{3}}{6} \qquad = \frac{5\sqrt{3}}{3}$$

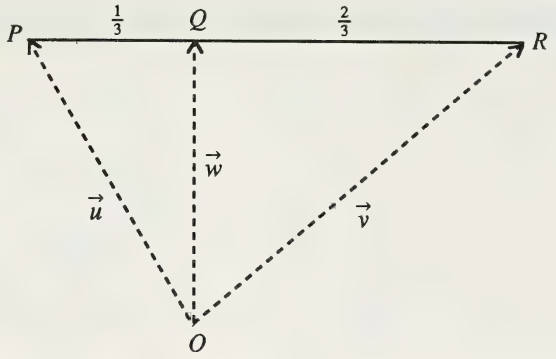
Therefore, $\vec{w} = \frac{5\sqrt{3}}{3}\vec{u} + \frac{5\sqrt{3}}{6}\vec{v}$.

3. Since Q divides PR in the ratio $PQ:PR = 1:3$ and $QR:PR = 2:3$, you can write the following:

$$\vec{w} = \frac{1}{3}\vec{v} + \frac{2}{3}\vec{u}$$

$$3\vec{w} = \vec{v} + 2\vec{u}$$

$$\therefore \vec{v} = 3\vec{w} - 2\vec{u}$$



4. First find $\overrightarrow{BC}$ and $\overrightarrow{CD}$ as linear combinations of the other vectors.

$$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$$

$$= -\overrightarrow{OB} + \overrightarrow{OC}$$

$$= -\left(\frac{7}{3}\overrightarrow{OC} - \frac{4}{3}\overrightarrow{OD}\right) + \overrightarrow{OC}$$

$$= \frac{-7}{3}\overrightarrow{OC} + \frac{4}{3}\overrightarrow{OD} + \overrightarrow{OC}$$

$$= -\frac{4}{3}\overrightarrow{OC} + \frac{4}{3}\overrightarrow{OD}$$

$$= \frac{4}{3}\left(\overrightarrow{OD} - \overrightarrow{OC}\right)$$

$\overrightarrow{CD} = \overrightarrow{CO} + \overrightarrow{OD}$

$= -\overrightarrow{OC} + \overrightarrow{OD}$

$= \overrightarrow{OD} - \overrightarrow{OC}$

$\overrightarrow{BC} = \frac{4}{3}\overrightarrow{CD}$

$\overrightarrow{BC}$ is a scalar multiple of $\overrightarrow{CD}$; therefore, B, C, and D are collinear.

Also, $\frac{\overrightarrow{BC}}{\overrightarrow{CD}}$ is $\frac{4}{3}$; therefore, $\frac{BC}{CD}$ is $\frac{4}{3}$.

5. $\overrightarrow{PT} = \frac{1}{2}\left(\overrightarrow{PQ} + \overrightarrow{PS}\right)$ (line division method)

$= \frac{1}{2}\left(\overrightarrow{PQ} + \overrightarrow{QR}\right)$ $\left(\overrightarrow{PS}\text{ and }\overrightarrow{QR}\text{ are equivalent vectors.}\right)$

$= \frac{1}{2}\left(\overrightarrow{PR}\right)$ $\left(\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}\right)$

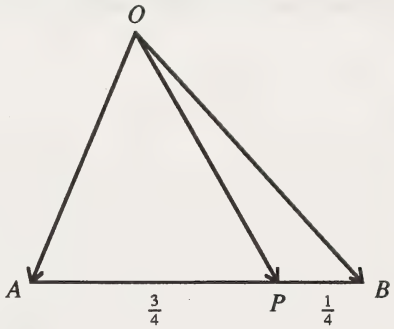
$= \overrightarrow{PU}$ (U is the midpoint of $\overrightarrow{PR}$.)

Conclusion:

Because T and U are the same point, the diagonals of the parallelogram bisect each other.

6. Draw the diagram.

$\overrightarrow{OP} = \frac{3}{4}\overrightarrow{OB} + \frac{1}{4}\overrightarrow{OA}$



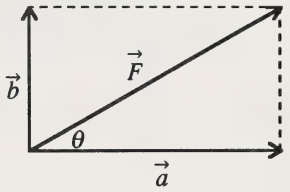
Extra Help

1. a. $|\vec{a}| = |\vec{F}|\cos\theta$

$= 50\text{ N}(\cos 60°)$

$= 50\text{ N}\left(\frac{1}{2}\right)$

$= 25\text{ N}$



$|\vec{b}| = |\vec{F}|\sin\theta$

$= 50\text{ N}(\sin 60°)$

$= 50\text{ N}\left(\frac{\sqrt{3}}{2}\right)$

$= 25\sqrt{3}\text{ N}$

b. $\left|\vec{a}\right| = 96\text{ N}(\cos 30°)$

$= 96\text{ N}\left(\frac{\sqrt{3}}{2}\right)$

$= 48\sqrt{3}\text{ N}$

$\left|\vec{b}\right| = 96\text{ N}(\sin 30°)$

$= 96\text{ N}\left(\frac{1}{2}\right)$

$= 48\text{ N}$

c. $\left|\vec{a}\right| = 150\text{ N}(\cos 45°)$

$= 150\text{ N}\left(\frac{1}{\sqrt{2}}\right)$

$= 150\text{ N}\left(\frac{\sqrt{2}}{2}\right)$

$= 75\sqrt{2}\text{ N}$

$\left|\vec{b}\right| = 150\text{ N}(\sin 45°)$

$= 150\text{ N}\left(\frac{1}{\sqrt{2}}\right)$

$= 150\text{ N}\left(\frac{\sqrt{2}}{2}\right)$

$= 75\sqrt{2}\text{ N}$

2. First draw the diagram.

$\vec{R} = \overrightarrow{F_1} + \overrightarrow{F_2}$

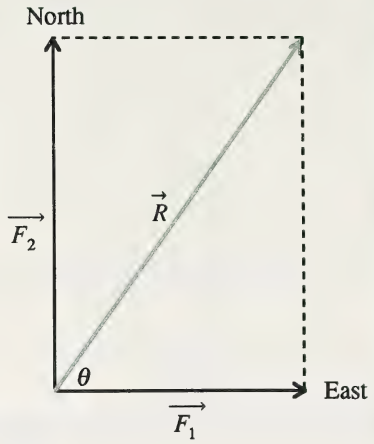
$\therefore \left|\vec{R}\right|^2 = 60^2 + 80^2$

$= 3600 + 6400$

$= 10\,000$

$\left|\vec{R}\right| = \sqrt{10\,000}$

$= 100$



Let the angle between the resultant and the easterly force be θ.

$\therefore \sin\theta = \frac{80}{100}$

$= 0.8$

$\theta \doteq 53°$

The magnitude of the resultant is 100 N, and the direction is approximately 53° with the horizontal.

3. $\overrightarrow{OP} = \frac{1}{6}\overrightarrow{OR} + \frac{5}{6}\overrightarrow{OQ}$

$\therefore \vec{u} = \frac{1}{6}\vec{w} + \frac{5}{6}\vec{v}$ $\left(\vec{u}\text{ as a linear combination of }\vec{w}\text{ and }\vec{v}\right)$

$\frac{5}{6}\vec{v} = \vec{u} - \frac{1}{6}\vec{w}$

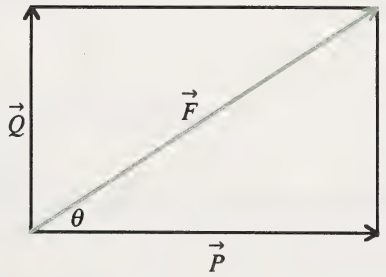
$\vec{v} = \frac{6}{5}\vec{u} - \frac{1}{5}\vec{w}$ $\left(\vec{v}\text{ as a linear combination of }\vec{u}\text{ and }\vec{w}\right)$

4. a. $\left|\vec{P}\right| = \left|\vec{F}\right| \cos\theta$

$= 75\text{ N}(\cos 25°)$

$= 75\text{ N}(0.9063)$

$\doteq 68\text{ N}$



$\left|\vec{Q}\right| = \left|\vec{F}\right| \sin\theta$

$= 75\text{ N}(\sin 25°)$

$= 75\text{ N}(0.4226)$

$\doteq 32\text{ N}$

b. $\left|\vec{P}\right| = \left|\vec{F}\right| \cos\theta$

$= 150\text{ N}(\cos 60°)$

$= 150\text{ N}(0.5)$

$= 75\text{ N}$

$\left|\vec{Q}\right| = \left|\vec{F}\right| \sin\theta$

$= 150\text{ N}(\sin 60°)$

$= 150\text{ N}(0.8660)$

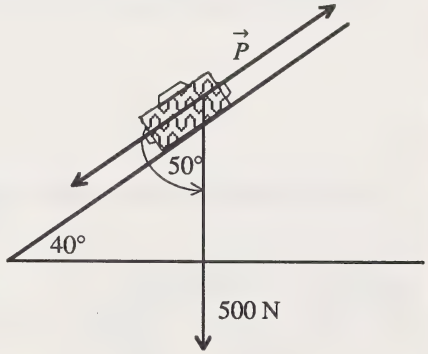
$\doteq 130\text{ N}$

5. Let the force be $\vec{P}$.

$\therefore \left|\vec{P}\right| = 500 \cos 50°$

$= 500(0.6428)$

$\doteq 321$



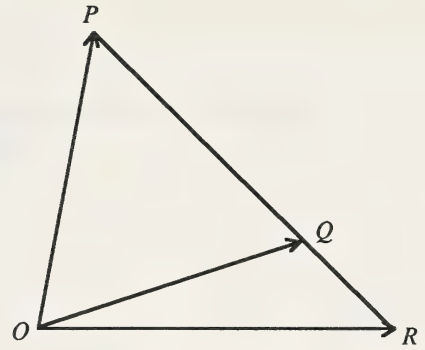
The force being applied to keep the trunk stationary on the ramp is approximately 321 N.

6. In order to show that P, Q, and R are collinear, you must show that $\overrightarrow{PQ}$ is a scalar multiple of $\overrightarrow{QR}$.

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PO} + \overrightarrow{OQ} \\ &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= \overrightarrow{OQ} - 3\overrightarrow{OQ} + 2\overrightarrow{OR} \\ &= 2\left(\overrightarrow{OR} - \overrightarrow{OQ}\right)\end{aligned}$$

$$\begin{aligned}\overrightarrow{QR} &= \overrightarrow{QO} + \overrightarrow{OR} \\ &= \overrightarrow{OR} - \overrightarrow{OQ}\end{aligned}$$

$$\therefore \overrightarrow{PQ} = 2\overrightarrow{QR}$$



Therefore, P, Q, and R are collinear and $PQ:QR = 2:1$.

Extensions

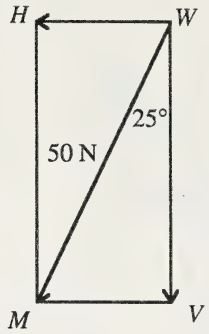
1. Draw a right triangle and label it to represent the given force and angle.

$$\cos 25° = \frac{\left|\overrightarrow{WV}\right|}{50}$$

$$\therefore \left|\overrightarrow{WV}\right| = 50 \cos 25°$$

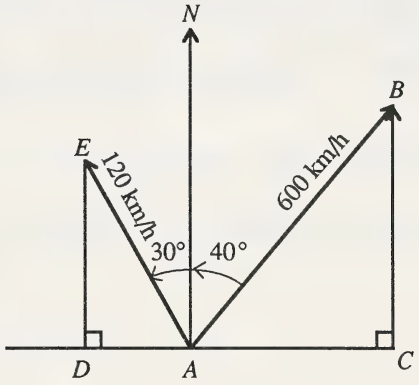
$$= 50(0.9063)$$

$$\doteq 45$$



The force that tends to lower the window is approximately 45 N.

2. Draw and label a diagram.



Let $\overrightarrow{DE}$ be the north component of the velocity of the wind and $\overrightarrow{CB}$ be the north component of air speed.

$\overrightarrow{AN}$ is the total velocity due north.

$\therefore \overrightarrow{AN} = \overrightarrow{DE} + \overrightarrow{CB}$ (1)

Consider ΔADE.

$$\sin 60° = \frac{\left|\overrightarrow{DE}\right|}{120}$$

$$\left|\overrightarrow{DE}\right| = 120 \sin 60°$$

$$= 120\frac{\sqrt{3}}{2}$$

$$= 60\sqrt{3}$$

$$\doteq 103.9 \quad (2)$$

Now consider ΔABC.

$$\sin 50° = \frac{\left|\overrightarrow{CB}\right|}{600}$$

$$\left|\overrightarrow{CB}\right| = 600 \sin 50°$$

$$= 600(0.7660)$$

$$\doteq 459.6 \quad (3)$$

Substitute (2) and (3) in (1).

$$\left|\overrightarrow{AN}\right| = 103.9 + 459.6$$

$$\doteq 54$$

In one hour the plane will fly approximately 564 km north.

3. From the information given draw and label a diagram.

$\overrightarrow{PQ}$ represents the velocity of the wind.

Therefore, the following may be stated using the sine law:

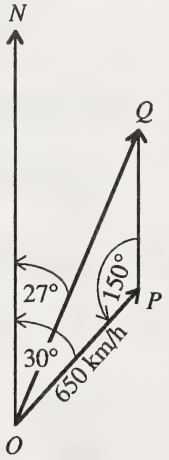
$$\frac{\left|\overrightarrow{PQ}\right|}{\sin 3°} = \frac{\left|\overrightarrow{OP}\right|}{\sin 27°}$$

$$\left|\overrightarrow{PQ}\right| = \frac{650 \sin 3°}{\sin 27°}$$

$$= \frac{650(0.0523)}{0.4540}$$

$$= 74.93$$

$$\doteq 75$$



The velocity of the wind with respect to the ground is approximately 75 km/h.

$\overrightarrow{OQ}$ is the velocity of the plane with respect to the ground.

$$\frac{\left|\overrightarrow{OQ}\right|}{\sin 150°} = \frac{\left|\overrightarrow{OP}\right|}{\sin 27°}$$

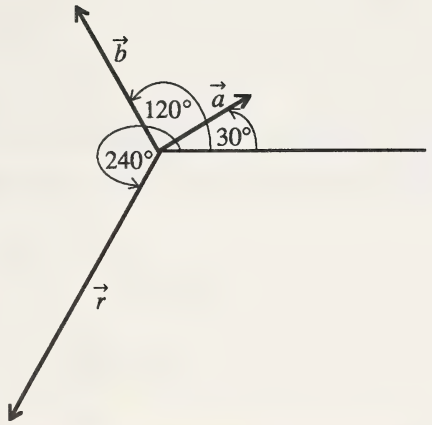
$$\left|\overrightarrow{OQ}\right| = \frac{650(0.5)}{0.4540}$$

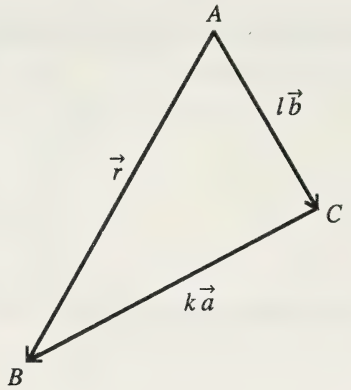
$$= \frac{325}{0.454}$$

$$\doteq 716$$

The velocity of the plane with respect to the ground is approximately 716 km/h.

4. First draw a diagram.





Let $\vec{r} = l\vec{b} + k\vec{a}$.

In ΔABC, $\left|\overrightarrow{AB}\right| = 6$, $\left|\overrightarrow{AC}\right| = 3|l|$, and $\left|\overrightarrow{CB}\right| = 2|k|$.

The angles in the triangle can be determined from the relationship between the angles of the given vectors.

From the given information $\angle ACB = 90°$, $\angle CBA = 30°$, and $\angle BAC = 60°$.

a. $\left|\overrightarrow{AC}\right| = \left|\overrightarrow{AB}\right| \sin 30°$

$$3|l| = 6 \sin 30°$$

$$|l| = 2\left(\frac{1}{2}\right)$$

$$= 1$$

$$\left|\overrightarrow{CB}\right| = \left|\overrightarrow{AB}\right| \cos 30°$$

$$2|k| = 6 \cos 30°$$

$$|k| = \frac{(6)(\sqrt{3})}{(2)(2)}$$

$$= \frac{3\sqrt{3}}{2}$$

From the diagram $k, l < 0$ since the direction of the vectors $\vec{a}$ and $\vec{b}$ in the triangle have been reversed.

$$\therefore \vec{r} = -\vec{b} - \frac{3\sqrt{3}}{2}\vec{a}$$

b. $\vec{r} = -\vec{b} - \frac{3\sqrt{3}}{2}\vec{a}$

$$\therefore \frac{3\sqrt{3}}{2}\vec{a} = -\left(\vec{r} + \vec{b}\right)$$

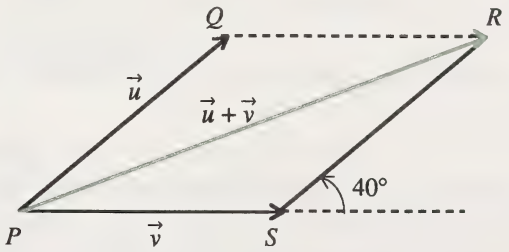
$$\vec{a} = \frac{-\left(\vec{r} + \vec{b}\right)\left(\frac{2}{1}\right)}{3\sqrt{3}}$$

$$= \frac{-2\sqrt{3}\left(\vec{r} + \vec{b}\right)}{9}$$

c. $\vec{b} = -\left(\vec{r} + \frac{3\sqrt{3}}{2}\vec{a}\right)$

5. First draw a diagram.

To find $\left|\vec{u} + \vec{v}\right|$ you must use the law of cosines. Note that $\angle PSR = 140°$, since it is supplementary to the 40° angle.



In ΔPSR $\left|\overrightarrow{PS}\right| = 14$ and $\left|\overrightarrow{SR}\right| = 10$.

$$\left|\overrightarrow{PR}\right|^2 = \left|\overrightarrow{PS}\right|^2 + \left|\overrightarrow{SR}\right|^2 - 2\left|\overrightarrow{PS}\right|\left|\overrightarrow{SR}\right| \cos 140°$$

$$= 14^2 + 10^2 - 2(14)(10) \cos 140°$$

$$= 196 + 100 - 280(-0.7660)$$

$$= 510.49$$

$$\therefore \left|\overrightarrow{PR}\right| \doteq 22.59$$

Thus, $\left|\vec{u} + \vec{v}\right| = \left|\overrightarrow{PR}\right| \doteq 22.59$ units.

To find $\angle RPS$, use the sine law.

$$\frac{\left|\overrightarrow{SR}\right|}{\sin \angle RPS} = \frac{\left|\overrightarrow{PR}\right|}{\sin 140°}$$

$$\sin \angle RPS = \frac{10 \sin 140°}{22.59}$$

$$= 0.2845$$

$$\therefore \angle RPS \doteq 17°$$

The angle between the resultant $\vec{u} + \vec{v}$ and $\vec{v}$ is approximately 17°.

6. $k\left|\vec{u}\right| = \left|\vec{w}\right| \cos 30°$

$$10k = 12\left(\frac{\sqrt{3}}{2}\right)$$

$$k = \frac{6\sqrt{3}}{10}$$

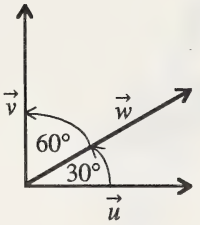
$$= \frac{3\sqrt{3}}{5}$$

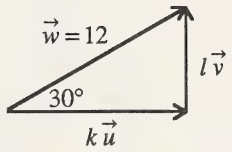
$l\left|\vec{v}\right| = \left|\vec{w}\right| \sin 30°$

$$12l = 12\left(\frac{1}{2}\right)$$

$$12l = 6$$

$$l = \frac{1}{2}$$





$\therefore \vec{w} = k\vec{u} + l\vec{v}$

$$= \frac{3\sqrt{3}}{5}\vec{u} + \frac{1}{2}\vec{v} \qquad \left(\vec{w} \text{ as a linear combination of } \vec{u} \text{ and } \vec{v}\right)$$

$$-\frac{1}{2}\vec{v} = -\vec{w} + \frac{3\sqrt{3}}{5}\vec{u}$$

$$\frac{1}{2}\vec{v} = \vec{w} - \frac{3\sqrt{3}}{5}\vec{u}$$

$$\vec{v} = 2\vec{w} - \frac{6\sqrt{3}}{5}\vec{u} \qquad \left(\vec{v} \text{ as a linear combination of } \vec{w} \text{ and } \vec{u}\right)$$

7. $$\frac{\left|\vec{w}\right|}{\sin R} = \frac{l\left|\vec{v}\right|}{\sin Q}$$

$$\frac{10}{\sin 120°} = \frac{4l}{\sin 30°}$$

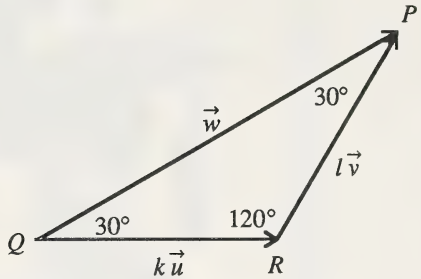
$$\frac{10}{\frac{\sqrt{3}}{2}} = \frac{4l}{\frac{1}{2}}$$

$$4l\left(\frac{\sqrt{3}}{2}\right) = 10\left(\frac{1}{2}\right)$$

$$4l = 5\left(\frac{2}{\sqrt{3}}\right)$$

$$l = \frac{10}{4\sqrt{3}}$$

$$= \frac{5\sqrt{3}}{6} \quad (1)$$



$$\frac{|\vec{w}|}{\sin R} = \frac{k|\vec{u}|}{\sin P}$$

$$\frac{10}{\sin 120°} = \frac{2k}{\sin 30°}$$

$$\frac{10}{\frac{\sqrt{3}}{2}} = \frac{2k}{\frac{1}{2}}$$

$$2k\left(\frac{\sqrt{3}}{2}\right) = 10\left(\frac{1}{2}\right)$$

$$\sqrt{3}k = 5$$

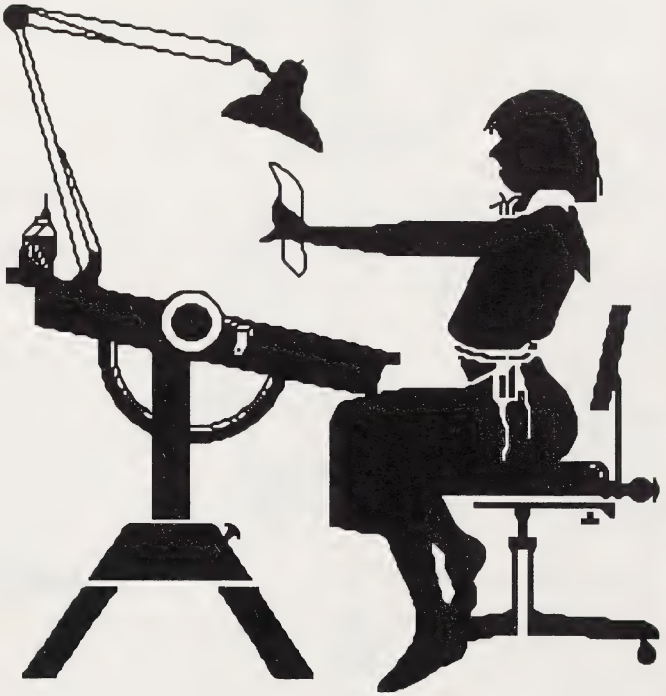
$$k = \frac{5}{\sqrt{3}}$$

$$= \frac{5\sqrt{3}}{3} \quad (2)$$

$$\vec{w} = k\vec{u} + l\vec{v} \quad (3)$$

Substituting (1) and (2) into (3) gives

$$\vec{w} = \frac{5\sqrt{3}}{3}\vec{u} + \frac{5\sqrt{3}}{6}\vec{v}.$$



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